A q-analog of the Racah polynomials and the q-algebra $SU_q(2)^*$

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Abstract

We study some q-analogues of the Racah polynomials and some of their applications in the theory of representation of quantum algebras.

1 Introduction

In the paper [6] an orthogonal polynomial family that generalizes the Racah coefficients or $6j$-symbols was introduced: the so-called Racah and q-Racah polynomials. These polynomials were in the top of the so-called Askey Scheme (see e.g. [14]) that contains all classical families of hypergeometric orthogonal polynomials. Some years later the same authors [7] introduced the celebrated Askey-Wilson polynomials. One of the important properties of these polynomials is that from them one can obtain all known families of hypergeometric polynomials and q-polynomials as particular cases or as limit cases (for a review on this see the nice survey [14]). The main tool in these two works was the hypergeometric and basic series, respectively. On the other hand, the authors of [21] (see also [20, Russian Edition]) considered the q-polynomials as the solution of a second order difference equation of hypergeometric-type on the non-linear lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$. In particular, they show that the solution of the hypergeometric-type equation can be expressed as certain basic series and, in such a way, they recovered the results by Askey & Wilson.

The interest of such polynomials increase after the appearance of the q-algebras and quantum groups [9, 10, 12, 16, 24]. However, from the first attends to built the q-analog of the Wigner-Racah formalism for the simplest quantum algebra $U_q(su(2))$ [13] (see also [1, 4, 17]) becomes clear that for obtaining the q-polynomials intimately connected with the q-analogues of the Racah and Clebsh Gordan coefficients, i.e., a q-analogue of the Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$ and the dual Hahn polynomials $w_n^c(x(s), a, b)_q$, respectively, it is better to use a different lattice —in fact the q-Racah polynomials $R_n^{\beta,\gamma}(x(s), N, \delta)_q$ introduced in [7] (see also [14]) were defined on the lattice $x(s) = q^{-s} + \delta q^{-N} q^s$ that depends not only of the variable s but also on the parameters of the polynomials—, namely,

$$x(s) = [s]_q [s+1]_q, \tag{1}$$

that only depends on s, where by $[s]_q$ we denote the q-numbers (in its symmetric form)

$$[s]_q = \frac{q^{s/2} - q^{-s/2}}{q^{1/2} - q^{-1/2}}, \quad \forall s \in \mathbb{C}. \tag{2}$$

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With this choice the q-Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ are proportional to the q-Racah coefficients (or $6j$-symbols) of the quantum algebra $U_q(su(2))$. A very nice and simple approach to $6j$-symbols has been recently developed in [23].

Moreover, this connection gives the possibility to a deeper study of the Wigner-Racah formalism (or the q-analogue of the quantum theory of angular momentum [25, 26, 27, 28]) for the quantum algebras $U_q(su(2))$ and $U_q(su(1,1))$ using the powerful and well-known theory of orthogonal polynomials on non-uniform lattices. On the other hand, using the q-analogue of the quantum theory of angular momentum [25, 26, 27, 28] we can obtain several results for the q-polynomials, some of which are non trivial from the point of view of the theory of orthogonal polynomials (see e.g. the nice surveys [15, 29]). In fact, in the present paper we present a detailed study of some q-analogues of the Racah polynomials on the lattice (1): the $u_n^{\alpha,\beta}(x(s),a,b)_q$ and the $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ as well as their connection with the q-Racah coefficients (or $6j$-symbols) of the quantum algebra $U_q(su(2))$ in order to establish which properties of the polynomials correspond to the $6j$-symbols and vice versa.

The structure of the paper is as follows: In section 2 we present some general results from the theory of orthogonal polynomials on the non-uniform lattices taken from [3, 20]. In Section 2.1 a detailed discussion of the Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ is presented, whereas in Section 2.2 the $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ are considered. In particular, a relation between these families is established. In section 3 the comparative analysis of such families and the $6j$-symbols of the quantum algebra $U_q(su(2))$ is developed which gives, on one hand, some information about the Racah coefficients and, on the other hand, allow us to give a group-theoretical interpretation of the Racah polynomials on the lattice (1). Finally, some comments and remarks about q-Racah polynomials and the quantum algebra $U_q(su(3))$ are included.

2 Some general properties of q-polynomials

We will start with some general properties of orthogonal hypergeometric polynomials on the non-uniform lattices [8, 20].

The hypergeometric polynomials are the polynomial solutions $P_n(x(s))_q$ of the second order linear difference equation of hypergeometric-type on the non-uniform lattice $x(s)$ (SODE)

$$\sigma(s)\frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla y(s)}{\nabla x(s)}+\tau(s)\frac{\Delta y(s)}{\Delta x(s)}+\lambda y(s)=0,\quad x(s)=c_1[q^s+q^{-s-\mu}]+c_3,\quad q^\mu=\frac{c_1}{c_2},$$
$$\nabla f(s)=f(s)-f(s-1),\qquad \Delta f(s)=f(s+1)-f(s), \tag{3}$$

or, equivalently

$$A_s y(s+1)+B_s y(s)+C_s y(s-1)+\lambda y(s)=0, \tag{4}$$

where

$$A_s=\frac{\sigma(s)+\tau(s)\Delta x(s-\frac{1}{2})}{\Delta x(s)\Delta x(s-\frac{1}{2})},\quad C_s=\frac{\sigma(s)}{\nabla x(s)\Delta x(s-\frac{1}{2})},\quad B_s=-(A_s+C_s).$$

Notice that $x(s)=x(-s-\mu)$.

In the following we will use the following notations[1] $P_n(s)_q:=P_n(x(s))_q$ and $\sigma(-s-\mu)=\sigma(s)+\tau(s)\Delta x(s-\frac{1}{2})$. With this notation the Eq. (3) becomes

$$\sigma(-s-\mu)\frac{\Delta P_n(s)_q}{\Delta x(s)}-\sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)}+\lambda_n\Delta x(s-\tfrac{1}{2})P_n(s)_q=0\,. \tag{5}$$

The polynomial solutions $P_n(s)_q$ of (3) can be obtained by the following Rodrigues-type formula [20, 22]

$$P_n(s)_q=\frac{B_n}{\rho(s)}\nabla^{(n)}\rho_n(s),\qquad \nabla^{(n)}:=\frac{\nabla}{\nabla x_1(s)}\frac{\nabla}{\nabla x_2(s)}\cdots\frac{\nabla}{\nabla x_n(s)}, \tag{6}$$

where $x_m(s)=x(s+\frac{m}{2})$,

$$\rho_n(s)=\rho(s+n)\prod_{m=1}^{n}\sigma(s+m), \tag{7}$$

[1] In the exponential lattice $x(s)=c_1q^{\pm s}+c_3$, so $\mu=\pm\infty$, therefore instead of using $\sigma(-s-\mu)$ one should use the equivalent function $\sigma(s)+\tau(s)\Delta x(s-\frac{1}{2})$.

and $\rho(s)$ is a solution of the Pearson-type equation $\Delta\left[\sigma(s)\rho(s)\right] = \tau(s)\rho(s)\Delta x(s-1/2)$, or equivalently,

$$\frac{\rho(s+1)}{\rho(s)} = \frac{\sigma(s)+\tau(s)\Delta x(s-\frac{1}{2})}{\sigma(s+1)} = \frac{\sigma(-s-\mu)}{\sigma(s+1)}. \tag{8}$$

Let us point out that the function ρ_n satisfy the equation $\Delta\left[\sigma(s)\rho_n(s)\right] = \tau_n(s)\rho_n(s)\Delta x_n(s-1/2)$, where $\tau_n(s)$ is given by

$$\tau_n(s) = \frac{\sigma(s+n)+\tau(s+n)\Delta x(s+n-\frac{1}{2})-\sigma(s)}{\Delta x_{n-1}(s)} = \frac{\sigma(-s-n-\mu)-\sigma(s)}{\Delta x_n(s-\frac{1}{2})} = \tau_n' x_n(s) + \tau_n(0). \tag{9}$$

being

$$\tau_n' = -\frac{\lambda_{2n+1}}{[2n+1]_q}, \qquad \tau_n(0) = \frac{\sigma(-s_n^\star-n-\mu)-\sigma(s_n^\star)}{x_n(s_n^\star+\frac{1}{2})-x_n(s_n^\star-\frac{1}{2})},$$

where $s_n^\star$ is the zero of the function $x_n(s)$, i.e., $x_n(s_n^\star)=0$.

From (6) follows an explicit formula for the polynomials P_n [20, Eq.(3.2.30)]

$$P_n(s)_q = B_n \sum_{m=0}^{n} \frac{[n]_q!(-1)^{m+n}}{[m]_q![n-m]_q!} \frac{\nabla x(s+m-\frac{n-1}{2})}{\prod\limits_{l=0}^{n}\nabla x(s+\frac{m-l+1}{2})} \frac{\rho_n(s-n+m)}{\rho(s)}, \tag{10}$$

where $[n]_q$ denotes the *symmetric q-numbers* (2) and the q-factorials are given by

$$[0]_q! := 1, \quad [n]_q! := [1]_q[2]_q\cdots[n]_q, \quad n\in\mathbb{N}.$$

It can be shown [8, 20, 22] that the most general polynomial solution of the q-hypergeometric equation (3) corresponds to

$$\sigma(s) = A\prod_{i=1}^{4}[s-s_i]_q = Cq^{-2s}\prod_{i=1}^{4}(q^s-q^{s_i}), \qquad A\cdot C\neq 0 \tag{11}$$

and has the form [22, Eq. (49a), page 240]

$$P_n(s)_q = D_n\; {}_4\phi_3\left(\begin{array}{c} q^{-n}, q^{2\mu+n-1+\sum_{i=1}^4 s_i}, q^{s_1-s}, q^{s_1+s+\mu} \\ q^{s_1+s_2+\mu}, q^{s_1+s_3+\mu}, q^{s_1+s_4+\mu} \end{array}; q\,,\, q\right), \tag{12}$$

where the normalizing factor D_n is given by ($\varkappa_q := q^{1/2}-q^{-1/2}$)

$$D_n = B_n\left(\frac{-A}{c_1 q^{\mu}\varkappa_q^5}\right)^n q^{-\frac{n}{2}(3s_1+s_2+s_3+s_4+\frac{3(n-1)}{2})}(q^{s_1+s_2+\mu};q)_n(q^{s_1+s_3+\mu};q)_n(q^{s_1+s_4+\mu};q)_n.$$

The basic hypergeometric series ${}_r\phi_p$ are defined by [14]

$${}_r\phi_p\left(\begin{array}{c} a_1,\dots,a_r \\ b_1,\dots,b_p \end{array}; q\,,\, z\right) = \sum_{k=0}^{\infty}\frac{(a_1;q)_k\cdots(a_r;q)_k}{(b_1;q)_k\cdots(b_p;q)_k}\frac{z^k}{(q;q)_k}\left[(-1)^k q^{\frac{k}{2}(k-1)}\right]^{p-r+1},$$

where $(a;q)_k = \prod_{m=0}^{k-1}(1-aq^m)$, is the q-analogue of the Pocchammer symbol.

In this paper we will deal with orthogonal q-polynomials and functions. It can be proven [20], by using the difference equation of hypergeometric-type (3), that if the boundary conditions $\sigma(s)\rho(s)x^k(s-1/2)\big|_{s=a,b} = 0$, for all $k\geq 0$, holds, then the polynomials $P_n(s)_q$ are orthogonal with respect to the weight function ρ, i.e.,

$$\sum_{s=a}^{b-1} P_n(s)_q P_m(s)_q \rho(s)\Delta x(s-1/2) = \delta_{nm}d_n^2, \quad s=a,a+1,\dots,b-1. \tag{13}$$

The squared norm in (13) is [20, Eq. (3.7.15)]

$$d_n^2 = (-1)^n A_{n,n}B_n^2\sum_{s=a}^{b-n-1}\rho_n(s)\Delta x_n(s-1/2), \tag{14}$$

where [20, page 66]

$$A_{n,k} = \frac{[n]_q!}{[n-k]_q!} \prod_{m=0}^{k-1} \left(-\frac{\lambda_{n+m}}{[n+m]_q} \right). \tag{15}$$

A simple consequence of the orthogonality is the three-term recurrence relation (TTRR)

$$x(s)P_n(s)_q = \alpha_n P_{n+1}(s)_q + \beta_n P_n(s)_q + \gamma_n P_{n-1}(s)_q, \tag{16}$$

where α_n, β_n and γ_n are given by

$$\alpha_n = \frac{a_n}{a_{n+1}}, \quad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}, \quad \gamma_n = \frac{a_{n-1}}{a_n} \frac{d_n^2}{d_{n-1}^2}, \tag{17}$$

being a_n and b_n the first and second coefficients in the power expansion of P_n, i.e., $P_n(s)_q = a_n x^n(s) + b_n x^{n-1}(s) + \cdots$. Substituting $s = a$ in (16) we find

$$\beta_n = \frac{x(a)P_n(a)_q - \alpha_n P_{n+1}(a)_q - \gamma_n P_{n-1}(a)_q}{P_n(a)_q}, \tag{18}$$

which is an alternative way for finding the coefficient β_n. Also we can use the expression [3, page 148]

$$\beta_n = \frac{[n]_q \tau_{n-1}(0)}{\tau'_{n-1}} - \frac{[n+1]_q \tau_n(0)}{\tau'_n} + c_3([n]_q + 1 - [n+1]_q).$$

To compute α_n (and β_n) we need the following formulas (see e.g. [3, page 147])

$$a_n = \frac{B_n A_{n,n}}{[n]_q!}, \qquad \frac{b_n}{a_n} = \frac{[n]_q \tau_{n-1}(0)}{\tau'_{n-1}} + c_3([n]_q - n). \tag{19}$$

The explicit expression of λ_n is [22, Eq. (52) page 232]

$$\begin{aligned} \lambda_n &= -\frac{A q^{\mu}}{c_1^2 (q^{1/2} - q^{-1/2})^4} [n]_q \, [s_1 + s_2 + s_3 + s_4 + 2\mu + n - 1]_q \\ &= -\frac{C\, q^{-n+1/2}}{c_1^2 (q^{1/2} - q^{-1/2})^2} (1 - q^n) \left(1 - q^{s_1+s_2+s_3+s_4+2\mu+n-1}\right), \end{aligned} \tag{20}$$

which can be obtained equating the largest powers of q^s in (5).

From the Rodrigues formula [19, 3, §5.6] follows that

$$\frac{\Delta P_n(s-\frac{1}{2})_q}{\Delta x(s-\frac{1}{2})} = \frac{-\lambda_n B_n}{\widetilde{B}_{n-1}} \widetilde{P}_{n-1}(s)_q, \tag{21}$$

where $\widetilde{P}_{n-1}$ denotes the polynomial orthogonal with respect to the weight function $\widetilde{\rho}(s) = \rho_1(s-\frac{1}{2})$. On the other hand, rewriting (3) as

$$\left(\sigma(s) \frac{\nabla}{\nabla x_1(s)} + \tau(s) I \right) \frac{\Delta}{\Delta x(s)} P_n(s)_q = -\lambda_n P_n(s)_q,$$

it can be substituted by the following two first-order difference equations

$$\frac{\Delta}{\Delta x(s)} P_n(s)_q = Q(s), \qquad \left(\sigma(s) \frac{\nabla}{\nabla x_1(s)} + \tau(s) I \right) Q(s) = -\lambda_n P_n(s)_q. \tag{22}$$

Using the fact that $\frac{\Delta}{\Delta x(s)} P_n(s)_q$ is a polynomial of degree $n-1$ on $x(s+1/2)$ (see [20, §3.1]) it follows that

$$\frac{\Delta}{\Delta x(s)} P_n(s)_q = C_n Q_{n-1}(s+\tfrac{1}{2}),$$

where C_n is a normalizing constant. Comparison with (21) implies that $Q(s)$ is the polynomial $\widetilde{P}_{n-1}$ orthogonal with respect to the function $\rho_1(s-\frac{1}{2})$ and $C_n = -\lambda_n B_n / \widetilde{B}_{n-1}$. Therefore, the second expression in (22) becomes

$$P_n(s)_q = \frac{B_n}{\widetilde{B}_{n-1}} \left(\sigma(s) \frac{\nabla}{\nabla x_1(s)} + \tau(s) I \right) \widetilde{P}_{n-1}(s+\tfrac{1}{2})_q. \tag{23}$$

The q-polynomials satisfy the following differentiation-type formula [19, 3, §5.6.1]

$$\sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)}=\frac{\lambda_n}{[n]_q\tau_n'}\left[\tau_n(s)P_n(s)_q-\frac{B_n}{B_{n+1}}P_{n+1}(s)_q\right]. \tag{24}$$

Then, using the explicit expression for the coefficient α_n, we find

$$\sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)}=\frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau_n'}P_n(s)_q-\frac{\alpha_n\lambda_{2n}}{[2n]_q}P_{n+1}(s)_q. \tag{25}$$

From the above equation using the identity $\Delta\frac{\nabla P_n(s)_q}{\nabla x(s)}=\frac{\Delta P_n(s)_q}{\Delta x(s)}-\frac{\nabla P_n(s)_q}{\nabla x(s)}$ as well as the SODE (5) we find

$$\sigma(-s-\mu)\frac{\Delta P_n(s)_q}{\Delta x(s)}=\frac{\lambda_n}{[n]_q\tau_n'}\left[\left(\tau_n(s)-[n]_q\tau_n'\Delta x(s-\tfrac{1}{2})\right)P_n(s)_q-\frac{B_n}{B_{n+1}}P_{n+1}(s)_q\right]. \tag{26}$$

To conclude this section we will introduce the following notation by Nikiforov and Uvarov [20, 22]. First we define another *q-analog of the Pocchammer symbols* [20, Eq. (3.11.1)]

$$(a|q)_k=\prod_{m=0}^{k-1}[a+m]_q=\frac{\widetilde{\Gamma}_q(a+k)}{\widetilde{\Gamma}_q(a)}=(-1)^k(q^a;q)_k(q^{1/2}-q^{-1/2})^{-k}q^{-\frac{k}{4}(k-1)-\frac{ka}{2}}, \tag{27}$$

where $\widetilde{\Gamma}_q(x)$ is the q-analog of the Γ function introduced in [20, Eq. (3.2.24)], and related to the classical q-Gamma function Γ_q by formula

$$\widetilde{\Gamma}_q(s)=q^{-\frac{(s-1)(s-2)}{4}}\Gamma_q(s)=q^{-\frac{(s-1)(s-2)}{4}}(1-q)^{1-s}\frac{(q;q)_\infty}{(q^s;q)_\infty},\quad 0<q<1.$$

Next we define the *q-hypergeometric function* ${}_r\mathrm{F}_p(\cdot|q,z)$

$${}_r\mathrm{F}_p\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_p\end{matrix}\;\middle|\;q\,,\,z\right)=\sum_{k=0}^{\infty}\frac{(a_1|q)_k(a_2|q)_k\cdots(a_r|q)_k}{(b_1|q)_k(b_2|q)_k\cdots(b_p|q)_k}\frac{z^k}{(1|q)_k}\left[\varkappa_q^{-k}q^{\frac{1}{4}k(k-1)}\right]^{p-r+1}, \tag{28}$$

where, as before, $\varkappa_q=q^{1/2}-q^{-1/2}$, and $(a|q)_k$ are given by (27). Notice that

$$\lim_{q\to1}{}_r\mathrm{F}_p\left(\begin{matrix}a_1,a_2,\dots,a_r\\ b_1,b_2,\dots,b_p\end{matrix}\;\middle|\;q\,,\,z\,\varkappa_q^{p-r+1}\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_r)_k}{(b_1)_k\cdots(b_p)_k}\frac{z^k}{k!}={}_r\mathrm{F}_p\left(\begin{matrix}a_1,a_2,\dots,a_r\\ b_1,b_2,\dots,b_p\end{matrix}\;\middle|\;z\right),$$

and

$${}_{p+1}\mathrm{F}_p\left(\begin{matrix}a_1,a_2,\dots,a_{p+1}\\ b_1,b_2,\dots,b_p\end{matrix}\;\middle|\;q\,,\,t\right)\Bigg|_{t\,=\,t_0}={}_{p+1}\varphi_p\left(\begin{matrix}q^{a_1},q^{a_2},\dots,q^{a_{p+1}}\\ q^{b_1},q^{b_2},\dots,q^{b_p}\end{matrix}\;\middle|\;q\,,\,z\right), \tag{29}$$

where $t_0=z\,q^{\frac{1}{2}(\sum_{i=1}^{p+1}a_i-\sum_{i=1}^{p}b_i-1)}$.

Using the above notation the polynomial solutions of (3) is [22, Eq. (49), page 232]

$$\begin{aligned}P_n(s)_q=B_n&\left(\frac{A}{c_1q^{-\frac{\mu}{2}}\varkappa_q^2}\right)^n(s_1+s_2+\mu|q)_n(s_1+s_3+\mu|q)_n\times\\ &(s_1+s_4+\mu|q)_n\;{}_4\mathrm{F}_3\left(\begin{matrix}-n,2\mu+n-1+\sum_{i=1}^4 s_i,s_1-s,s_1+s+\mu\\ s_1+s_2+\mu,s_1+s_3+\mu,s_1+s_4+\mu\end{matrix}\;\middle|\;q\,,\,1\right).\end{aligned} \tag{30}$$

2.1 The q-Racah polynomials

Here we will consider the q-Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ on the lattice $x(s)=[s]_q[s+1]_q$ introduced in [2, 17, 20]. For this lattice one has

$$c_1=q^{\frac{1}{2}}\varkappa_q^{-2},\quad \mu=1,\quad c_3=-(q^{\frac{1}{2}}+q^{-\frac{1}{2}})\varkappa_q^{-2}. \tag{31}$$

Let chose σ in (11) as

$$\sigma(s)=-\frac{q^{-2s}}{\varkappa_q^4 q^{\frac{\alpha+\beta}{2}}}(q^s-q^a)(q^s-q^{-b})(q^s-q^{\beta-a})(q^s-q^{b+\alpha})=[s-a]_q[s+b]_q[s+a-\beta]_q[b+\alpha-s]_q,$$

i.e., $s_1=a, s_2=-b, s_3=\beta-a, s_4=b+\alpha$, $C=-q^{-\frac{1}{2}(\alpha+\beta)}\varkappa_q^{-4}$, $A=-1$, and let $B_n=(-1)^n/[n]_q!$. Here, as before, $\varkappa_q=q^{1/2}-q^{-1/2}$. Now from (20) we find

$$\lambda_n=q^{-\frac{1}{2}(\alpha+\beta+2n+1)}\varkappa_q^{-2}(1-q^n)(1-q^{\alpha+\beta+n+1})=[n]_q[n+\alpha+\beta+1]_q.$$

To obtain $\tau_n(s)$ we use (9). In this case $x_n(s)=[s+n/2]_q[s+n/2+1]_q$, then, choosing $s_n^\star=-n/2$, we get

$$\tau_n(s)=\tau_n' x_n(s)+\tau_n(0),\quad \tau_n'=-[2n+\alpha+\beta+2]_q,\quad \tau_n(0)=\sigma(-n/2-1)-\sigma(-n/2). \tag{32}$$

Taking into account that $\tau(s)=\tau_0(s)$, we obtain the corresponding function $\tau(s)$

$$\tau(s)=-[2+\alpha+\beta]_q x(s)+\sigma(-1)-\sigma(0).$$

2.1.1 The orthogonality and the norm d_n^2

A solution of the Pearson-type difference equation (8) is

$$\rho(s)=\frac{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(s+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(b-s)}.$$

Since $\sigma(a)\rho(a)=\sigma(b)\rho(b)=0$, then the q-Racah polynomials satisfy the orthogonality relation

$$\sum_{s=a}^{b-1}u_n^{\alpha,\beta}(x(s),a,b)_q u_m^{\alpha,\beta}(x(s),a,b)_q\rho(s)[2s+1]_q=0,\qquad n\neq m,$$

with the restrictions $-\frac{1}{2}<a\leq b-1, \alpha>-1, -1<\beta<2a+1$. Let us now compute the square of the norm d_n^2. From (7) and (15) follow

$$\rho_n(s)=\frac{\widetilde{\Gamma}_q(s+n+a+1)\widetilde{\Gamma}_q(s+n-a+\beta+1)\widetilde{\Gamma}_q(s+n+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(b-s-n)},$$

$$A_{n,n}=[n]_q!(-1)^n\frac{\widetilde{\Gamma}_q(\alpha+\beta+2n+1)}{\widetilde{\Gamma}_q(\alpha+\beta+n+1)}\quad\Rightarrow\quad \Lambda_n:=(-1)^nA_{n,n}B_n^2=\frac{\widetilde{\Gamma}_q(\alpha+\beta+2n+1)}{[n]_q!\widetilde{\Gamma}_q(\alpha+\beta+n+1)}.$$

Taking into account that $\nabla x_{n+1}(s)=[2s+n+1]_q$, using (14), and the identity

$$\widetilde{\Gamma}_q(A-s)=\frac{\widetilde{\Gamma}_q(A)(-1)^s}{(1-A|q)_s}, \tag{33}$$

we have

$$d_n^2=\Lambda_n\sum_{s=a}^{b-n-1}\frac{\widetilde{\Gamma}_q(s+n+a+1)\widetilde{\Gamma}_q(s+n-a+\beta+1)\widetilde{\Gamma}_q(s+n+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(b-s-n)[2s+n+1]_q^{-1}}$$

$$=\Lambda_n\sum_{s=0}^{b-a-n-1}\frac{\widetilde{\Gamma}_q(s+n+2a+1)\widetilde{\Gamma}_q(s+n+\beta+1)\widetilde{\Gamma}_q(s+n+\alpha+b+a+1)\widetilde{\Gamma}_q(b-a+\alpha-s)}{\widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(s+b+a+1)\widetilde{\Gamma}_q(s+2a-\beta+1)\widetilde{\Gamma}_q(b-a-s-n)[2s+2a+n+1]_q^{-1}}$$

$$=\frac{\widetilde{\Gamma}_q(\alpha+\beta+2n+1)\widetilde{\Gamma}_q(2a+n+1)\widetilde{\Gamma}_q(n+\beta+1)\widetilde{\Gamma}_q(a+b+n+\alpha+1)\widetilde{\Gamma}_q(b+\alpha-a)}{[n]_q!\widetilde{\Gamma}_q(\alpha+\beta+n+1)\widetilde{\Gamma}_q(a+b+1)\widetilde{\Gamma}_q(2a-\beta+1)\widetilde{\Gamma}_q(b-a-n)}\times$$

$$\sum_{s=0}^{b-a-n-1}\frac{(n+2a+1,n+\beta+1,n+a+\alpha+b+1,1-b+a+n|q)_s}{(1,a+b+1,2a-\beta+1,1-b+a-\alpha|q)_s}[2s+2a+n+1]_q.$$

In the following we denote by S_n the sum in the last expression. If we now use that $(a|q)_n = (-1)^n (q^a;q)_n q^{-\frac{n}{4}(n+2a-1)} \varkappa_q^{-n}$, as well as the identity

$$[2s+2a+n+1]_q = q^{-s}[2a+n+1]_q \frac{(q^{a+\frac{n+1}{2}+1};q)(-q^{a+\frac{n+1}{2}+1};q)_s}{(q^{a+\frac{n+1}{2}};q)(-q^{a+\frac{n+1}{2}};q)},$$

we obtain

$$S_n = \sum_{s=0}^{b-a-n-1} \frac{(q^{2a+n+1}, q^{n+\beta+1}, q^{n+\alpha+b+a+1}, q^{1-b+a+n}, q^{\frac{1}{2}(2a+n+3)}, -q^{\frac{1}{2}(2a+n+3)};q)_s}{(q, q^{a+b+1}, q^{2a-\beta+1}, q^{1-b-\alpha+a}, q^{\frac{1}{2}(2a+n+1)}, -q^{\frac{1}{2}(2a+n+1)};q)_s [2a+n+1]_q^{-1}} q^{-s(1+2n+\beta+\alpha)}$$

$$= [2a+n+1]_q \, {}_6\phi_5\left(\begin{matrix} q^{2a+n+1}, q^{n+\beta+1}, q^{n+a+\alpha+b+1}, q^{1-b+a+n}, q^{\frac{1}{2}(2a+n+3)}, -q^{\frac{1}{2}(2a+n+3)} \\ q^{a+b+1}, q^{2a-\beta+1}, q^{1-b-\alpha+a}, q^{\frac{1}{2}(2a+n+1)}, -q^{\frac{1}{2}(2a+n+1)} \end{matrix} \,\middle|\, q, q^{-1-2n-\beta-\alpha} \right).$$

But the above ${}_6\phi_5$ series is a very-well-poised ${}_6\phi_5$ basic series and therefore by using the summation formula [11, Eq. (II.21) page 238]

$${}_6\varphi_5\left(\begin{matrix} a, \; qa^{1/2}, \; -qa^{1/2}, \; b, \; c, \; q^{-k} \\ a^{1/2}, \; -a^{1/2}, aq/b, \; aq/c, aq^{k+1} \end{matrix} \,\middle|\, q, \frac{aq^{k+1}}{bc} \right) = \frac{(aq, aq/bc;q)_k}{(aq/b, aq/c;q)_k},$$

with $k = b-a-n-1$, $a = q^{2a+n+1}$, $b = q^{n+\beta+1}$, $c = q^{n+a+\alpha+b+1}$, we obtain

$$S_n = [2a+n+1]_q \frac{(q^{2a+n+2}, q^{-n+a-b-\alpha-\beta};q)_{b-a-n-1}}{(q^{2a-\beta+1}, q^{a-b-\alpha+1};q)_{b-a-n-1}}$$
$$= [2a+n+1]_q \frac{(2a+n+2|q)_{b-a-n-1}(-n+a-b-\alpha-\beta|q)_{b-a-n-1}}{(2a-\beta+1|q)_{b-a-n-1}(a-b-\alpha+1|q)_{b-a-n-1}}.$$

Finally, using (33) and (27)

$$S_n = [2a+n+1]_q \frac{\widetilde{\Gamma}_q(a+b+1)\widetilde{\Gamma}_q(2a-\beta+1)\widetilde{\Gamma}_q(b-a+\alpha+\beta+n+1)\widetilde{\Gamma}_q(\alpha+n+1)}{\widetilde{\Gamma}_q(n+2a+2)\widetilde{\Gamma}_q(b+a-\beta-n)\widetilde{\Gamma}_q(\alpha+\beta+2n+2)\widetilde{\Gamma}_q(b-a+\alpha)},$$

thus

$$d_n^2 = \frac{\widetilde{\Gamma}_q(\alpha+\beta+2n+1)\widetilde{\Gamma}_q(2a+n+1)\widetilde{\Gamma}_q(n+\beta+1)\widetilde{\Gamma}_q(a+b+n+\alpha+1)\widetilde{\Gamma}_q(b+\alpha-a)}{[n]_q!\widetilde{\Gamma}_q(\alpha+\beta+n+1)\widetilde{\Gamma}_q(a+b+1)\widetilde{\Gamma}_q(2a-\beta+1)\widetilde{\Gamma}_q(b-a-n)} S_n$$
$$= \frac{\widetilde{\Gamma}_q(\alpha+n+1)\widetilde{\Gamma}_q(\beta+n+1)\widetilde{\Gamma}_q(b-a+\alpha+\beta+n+1)\widetilde{\Gamma}_q(a+b+\alpha+n+1)}{[\alpha+\beta+2n+1]_q\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(\alpha+\beta+n+1)\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(a+b-\beta-n)}.$$

2.1.2 The hypergeometric representation

From formula (12) and (30) the following two equivalent hypergeometric representations hold

$$u_n^{\alpha,\beta}(x(s),a,b)_q = \frac{q^{-\frac{n}{2}(2a+\alpha+\beta+n+1)}(q^{a-b+1};q)_n(q^{\beta+1};q)_n(q^{a+b+\alpha+1};q)_n}{\varkappa_q^{2n}(q;q)_n} \times$$
$${}_4\varphi_3\left(\begin{matrix} q^{-n}, q^{\alpha+\beta+n+1}, q^{a-s}, q^{a+s+1} \\ q^{a-b+1}, q^{\beta+1}, q^{a+b+\alpha+1} \end{matrix} \,\middle|\, q\,,\, q \right), \tag{34}$$

and

$$u_n^{\alpha,\beta}(x(s),a,b)_q = \frac{(a-b+1|q)_n(\beta+1|q)_n(a+b+\alpha+1|q)_n}{[n]_q!} \times$$
$${}_4\mathrm{F}_3\left(\begin{matrix} -n, \alpha+\beta+n+1, a-s, a+s+1 \\ a-b+1, \beta+1, a+b+\alpha+1 \end{matrix} \,\middle|\, q\,,\, 1 \right). \tag{35}$$

Using the Sears transformation formula [11, Eq. (III.15)] we obtain the equivalent formulas

$$u_n^{\alpha,\beta}(x(s),a,b)_q = \frac{q^{-\frac{n}{2}(-2b+\alpha+\beta+n+1)}(q^{a-b+1};q)_n(q^{\alpha+1};q)_n(q^{\beta-a-b+1};q)_n}{\varkappa_q^{2n}(q;q)_n} \times$$
$${}_4\varphi_3\left(\begin{matrix} q^{-n}, q^{\alpha+\beta+n+1}, q^{-b-s}, q^{-b+s+1} \\ q^{a-b+1}, q^{\alpha+1}, q^{-a-b+\beta+1} \end{matrix} \,\middle|\, q\,,\, q \right), \tag{36}$$

and

$$u_n^{\alpha,\beta}(x(s),a,b)_q = \frac{(a-b+1|q)_n(\alpha+1|q)_n(-a-b+\beta+1|q)_n}{[n]_q!}\times \\ {}_4\mathrm{F}_3\left(\begin{array}{c} -n,\alpha+\beta+n+1,-b-s,-b+s+1 \\ a-b+1,\alpha+1,-a-b+\beta+1 \end{array}\,\middle|\, q\,,\,1\right). \tag{37}$$

Remark: From the above formulas follow that the polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ are multiples of the standard q-Racah polynomials $R_n(\mu(q^{b+s});q^\alpha,q^\beta,q^{a-b},q^{-a-b}|q)$.

From the above hypergeometric representations also follow the values

$$\begin{aligned} u_n^{\alpha,\beta}(x(a),a,b)_q =& \frac{(a-b+1|q)_n(\beta+1|q)_n(a+b+\alpha+1|q)_n}{[n]_q!} \\ =& \frac{(q^{a-b+1};q)_n(q^{\beta+1};q)_n(q^{a+b+\alpha+1};q)_n}{q^{\frac{n}{2}(2a+\alpha+\beta+n+1)}\varkappa_q^{2n}(q;q)_n}, \end{aligned} \tag{38}$$

$$\begin{aligned} u_n^{\alpha,\beta}(x(b-1),a,b)_q =& \frac{(a-b+1|q)_n(\alpha+1|q)_n(-a-b+\beta+1|q)_n}{[n]_q!} \\ =& \frac{(q^{a-b+1};q)_n(q^{\alpha+1};q)_n(q^{\beta-a-b+1};q)_n}{q^{\frac{n}{2}(-2b+\alpha+\beta+n+1)}\varkappa_q^{2n}(q;q)_n}. \end{aligned} \tag{39}$$

The formula (10) leads to the following explicit formula[2]

$$\begin{aligned} u_n^{\alpha,\beta}(x(s),a,b)_q =& \frac{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(b-s)}{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(s+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)}\times \\ & \sum_{k=0}^{n}\frac{(-1)^k[2s+2k-n+1]_q\widetilde{\Gamma}_q(s+k+a+1)\widetilde{\Gamma}_q(2s+k-n+1)}{\widetilde{\Gamma}_q(k+1)\widetilde{\Gamma}_q(n-k+1)\widetilde{\Gamma}_q(2s+k+2)\widetilde{\Gamma}_q(s-n+k-a+1)}\times \\ & \frac{\widetilde{\Gamma}_q(s+k-a+\beta+1)\widetilde{\Gamma}_q(s+k+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s+n-k)}{\widetilde{\Gamma}_q(s-n+k+b+1)\widetilde{\Gamma}_q(s-n+k+a-\beta+1)\widetilde{\Gamma}_q(b-s-k)}, \end{aligned} \tag{40}$$

from where follows

$$\begin{aligned} u_n^{\alpha,\beta}(x(a),a,b)_q =& \frac{(-1)^n\widetilde{\Gamma}_q(b-a)\widetilde{\Gamma}_q(\beta+n+1)\widetilde{\Gamma}_q(b+a+\alpha+n+1)}{[n]!\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(\beta+1)\widetilde{\Gamma}_q(b+a+\alpha+1)}, \\ u_n^{\alpha,\beta}(x(b-1),a,b)_q =& \frac{\widetilde{\Gamma}_q(b-a)\widetilde{\Gamma}_q(\alpha+n+1)\widetilde{\Gamma}_q(b+a-\beta)}{[n]!\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(\alpha+1)\widetilde{\Gamma}_q(b+a-\beta-n)}, \end{aligned} \tag{41}$$

that coincide with the values (38) and (39) obtained before.

From the hypergeometric representation the following symmetry property follows

$$u_n^{\alpha,\beta}(x(s),a,b)_q = u_n^{-b-a+\beta,b+a+\alpha}(x(s),a,b)_q.$$

Finally, notice that from (34) (or (36)) follows that $u_n^{\alpha,\beta}(x(s),a,b)_q$ is a polynomial of degree n on $x(s)=[s]_q[s+1]_q$. In fact,

$$(q^{a-s};q)_k(q^{a+s+1};q)_k = (-1)^k q^{k(a+\frac{k+1}{2})}\prod_{l=0}^{k-1}\left(\frac{x(s)-c_3}{c_1} - q^{-\frac{1}{2}}(q^{a+l+\frac{1}{2}}+q^{-a-l-\frac{1}{2}})\right),$$

where c_1 and c_3 are given in (31).

2.1.3 Three-term recurrence relation and differentiation formulas

To derive the coefficients of the TTRR (16) we use (17) and (18). Using (19) and (17), we obtain

$$a_n = \frac{\widetilde{\Gamma}_q(\alpha+\beta+2n+1)}{[n]_q!\widetilde{\Gamma}_q(\alpha+\beta+n+1)}, \quad \alpha_n = \frac{[n+1]_q[\alpha+\beta+n+1]_q}{[\alpha+\beta+2n+1]_q[\alpha+\beta+2n+2]_q}.$$

[2] Obviously the formulas (34) and (36) also give equivalent explicit formulas.

Table 1: Main data of the q-Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$

$P_n(s)$	$u_n^{\alpha,\beta}(x(s),a,b)_q\,, \quad x(s)=[s]_q[s+1]_q$
(a,b)	$[a,b-1]$
$\rho(s)$	$\dfrac{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(s+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(b-s)}$ $-\frac{1}{2}<a\le b-1, \alpha>-1, -1<\beta<2a+1$
$\sigma(s)$	$[s-a]_q[s+b]_q[s+a-\beta]_q[b+\alpha-s]_q$
$\sigma(-s-1)$	$[s+a+1]_q[b-s-1]_q[s-a+\beta+1]_q[b+\alpha+s+1]_q$
$\tau(s)$	$[\alpha+1]_q[a]_q[a-\beta]_q+[\beta+1]_q[b]_q[b+\alpha]_q-[\alpha+1]_q[\beta+1]_q-[\alpha+\beta+2]_q x(s)$
$\tau_n(s)$	$-[\alpha+\beta+2n+2]_q x(s+\frac{n}{2})+[a+\frac{n}{2}+1]_q[b-\frac{n}{2}-1]_q[\beta+\frac{n}{2}+1-a]_q[b+\alpha+\frac{n}{2}+1]_q$ $-[a+\frac{n}{2}]_q[b-\frac{n}{2}]_q[\beta+\frac{n}{2}-a]_q[b+\alpha+\frac{n}{2}]_q$
λ_n	$[n]_q[\alpha+\beta+n+1]_q$
B_n	$\dfrac{(-1)^n}{[n]_q!}$
d_n^2	$\dfrac{\widetilde{\Gamma}_q(\alpha+n+1)\widetilde{\Gamma}_q(\beta+n+1)\widetilde{\Gamma}_q(b-a+\alpha+\beta+n+1)\widetilde{\Gamma}_q(a+b+\alpha+n+1)}{[\alpha+\beta+2n+1]_q\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(\alpha+\beta+n+1)\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(a+b-\beta-n)}$
$\rho_n(s)$	$\dfrac{\widetilde{\Gamma}_q(s+n+a+1)\widetilde{\Gamma}_q(s+n-a+\beta+1)\widetilde{\Gamma}_q(s+n+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(b-s-n)}$
a_n	$\dfrac{\widetilde{\Gamma}_q[\alpha+\beta+2n+1]_q}{[n]_q!\widetilde{\Gamma}_q[\alpha+\beta+n+1]_q}$
α_n	$\dfrac{[n+1]_q[\alpha+\beta+n+1]_q}{[\alpha+\beta+2n+1]_q[\alpha+\beta+2n+2]_q}$
β_n	$[a]_q[a+1]_q-\dfrac{[\alpha+\beta+n+1]_q[a-b+n+1]_q[\beta+n+1]_q[a+b+\alpha+n+1]_q}{[\alpha+\beta+2n+1]_q[\alpha+\beta+2n+2]_q}$ $+\dfrac{[\alpha+n]_q[b-a+\alpha+\beta+n]_q[a+b-\beta-n]_q[n]_q}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+1]_q}$
γ_n	$\dfrac{[a+b+\alpha+n]_q[a+b-\beta-n]_q[\alpha+n]_q[\beta+n]_q[b-a+\alpha+\beta+n]_q[b-a-n]_q}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+1]_q}$

To find γ_n we use (17)

$$\gamma_n=\frac{[a+b+\alpha+n]_q[a+b-\beta-n]_q[\alpha+n]_q[\beta+n]_q[b-a+\alpha+\beta+n]_q[b-a-n]_q}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+1]_q}.$$

To compute β_n we use (18)

$$\begin{aligned}\beta_n= &\quad x(a)-\alpha_n\frac{u_{n+1}^{\alpha,\beta}(x(a),a,b)_q}{u_n^{\alpha,\beta}(x(a),a,b)_q}-\gamma_n\frac{u_{n-1}^{\alpha,\beta}(x(a),a,b)_q}{u_n^{\alpha,\beta}(x(a),a,b)_q}\\ = &\quad [a]_q[a+1]_q-\frac{[\alpha+\beta+n+1]_q[a-b+n+1]_q[\beta+n+1]_q[a+b+\alpha+n+1]_q}{[\alpha+\beta+2n+1]_q[\alpha+\beta+2n+2]_q}\\ &+\frac{[\alpha+n]_q[b-a+\alpha+\beta+n]_q[a+b-\beta-n]_q[n]_q}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+1]_q}.\end{aligned}$$

The differentiation formulas (21) and (23) yield

$$\frac{\Delta u_n^{\alpha,\beta}(x(s),a,b)_q}{\Delta x(s)}=[\alpha+\beta+n+1]_q u_{n-1}^{\alpha+1,\beta+1}(x(s+\tfrac{1}{2}),a+\tfrac{1}{2},b-\tfrac{1}{2})_q, \tag{42}$$

$$
\begin{aligned}
-[n]_q[2s+1]_q u_n^{\alpha,\beta}(x(s),a,b)_q =&\sigma(-s-1)u_{n-1}^{\alpha+1,\beta+1}(x(s+\tfrac{1}{2}),a+\tfrac{1}{2},b-\tfrac{1}{2})_q \\
&- \sigma(s)u_{n-1}^{\alpha+1,\beta+1}(x(s-\tfrac{1}{2}),a+\tfrac{1}{2},b-\tfrac{1}{2})_q,
\end{aligned}
\tag{43}
$$

respectively. Finally, formulas (24) (or (25)) and (26) lead to the *differentiation formulas*

$$
\sigma(s)\frac{\nabla u_n^{\alpha,\beta}(x(s),a,b)_q}{[2s]_q} = -\frac{[\alpha+\beta+n+1]_q}{[\alpha+\beta+2n+2]_q}\left[\tau_n(s)u_n^{\alpha,\beta}(x(s),a,b)_q + [n+1]_q u_{n+1}^{\alpha,\beta}(x(s),a,b)_q\right],
\tag{44}
$$

$$
\begin{aligned}
\sigma(-s-1)\frac{\Delta u_n^{\alpha,\beta}(x(s),a,b)_q}{[2s+2]_q} &= -\frac{[\alpha+\beta+n+1]_q}{[\alpha+\beta+2n+2]_q}\times \\
&\left[(\tau_n(s)+[n]_q[\alpha+\beta+2n+2]_q[2s+1]_q)u_n^{\alpha,\beta}(x(s),a,b)_q + [n+1]_q u_{n+1}^{\alpha,\beta}(x(s),a,b)_q\right],
\end{aligned}
\tag{45}
$$

where $\tau_n(s)$ is given in (32).

2.1.4 The duality of the Racah polynomials

In this section we will discuss the duality property of the q-Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$. We will follow [20, pages 38-39]. First of all, notice that the orthogonal relation (13) for the Racah polynomials can be written in the form

$$
\sum_{t=0}^{N-1} C_{tn}C_{tm} = \delta_{n,m}, \quad C_{tn} = \frac{u_n^{\alpha,\beta}(x(t+a),a,b)_q\sqrt{\rho(t+a)\Delta x(t+a-1/2)}}{d_n}, \quad N=b-a,
$$

where $\rho(s)$ and d_n are the weight function and the norm of the q-Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$, respectively. The above relation can be understood as the orthogonality property of the orthogonal matrix $C=(C_{tn})_{t,n=0}^{N-1}$ by its first index. If we now use the orthogonality of C by the second index we get

$$
\sum_{n=0}^{N-1} C_{tn}C_{t'n} = \delta_{t,t'}, \quad N=b-a,
$$

that leads to the dual orthogonality relation for the q-Racah polynomials

$$
\sum_{n=0}^{N-1} u_n^{\alpha,\beta}(x(s),a,b)_q u_n^{\alpha,\beta}(x(s'),a,b)_q \frac{1}{d_n^2} = \frac{1}{\rho(s)\Delta x(s-1/2)}\delta_{s,s'}.
\tag{46}
$$

The next step is to identify the functions $u_n^{\alpha,\beta}(x(s),a,b)_q$ as polynomials on some lattice $x(n)$. Before starting let us mention that from the representation (34) and the identity

$$
(q^{-n};q)_k(q^{\alpha+\beta+n+1};q)_k = \prod_{l=0}^{k-1}\left(1+q^{\alpha+\beta+2l+1} - q^{\frac{\alpha+\beta+1}{2}+l}\left(\varkappa_q^2 x(t) + q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right)\right),
$$

where $x(t)=[t]_q[t+1]_q=[n+\frac{\alpha+\beta}{2}]_q[n+\frac{\alpha+\beta}{2}+1]_q$, follows that $u_n^{\alpha,\beta}(x(s),a,b)_q$ also constitutes a polynomial of degree $s-a$ (for $s=a,a+1,\dots,b-a-1$) on $x(t)$ with $t=n+\frac{\alpha+\beta}{2}$.

Let us now define the polynomials —compare with the definition of the Racah polynomials (35)—

$$
\begin{aligned}
\mathrm{u}_k^{\alpha',\beta'}(x(t),a',b')_q =& \frac{(-1)^k\widetilde{\Gamma}_q(b'-a')\widetilde{\Gamma}_q(\beta'+k+1)\widetilde{\Gamma}_q(b'+a'+\alpha'+k+1)}{[k]!\widetilde{\Gamma}_q(b'-a'-k)\widetilde{\Gamma}_q(\beta'+1)\widetilde{\Gamma}_q(b'+a'+\alpha'+1)}\times \\
&{}_4\mathrm{F}_3\left(\begin{array}{c} -k,\alpha'+\beta'+k+1,a'-t,a'+t+1 \\ a'-b'+1,\beta'+1,a'+b'+\alpha'+1\end{array}\,\middle|\, q\,,\,1\right),
\end{aligned}
\tag{47}
$$

where

$$k = s - a, \quad t = n + \frac{\alpha+\beta}{2}, \quad a' = \frac{\alpha+\beta}{2}, \quad b' = b - a + \frac{\alpha+\beta}{2}, \quad \alpha' = 2a - \beta, \quad \beta' = \beta. \quad (48)$$

Obviously they are polynomials of degree $k = s-a$ on the lattice $x(t)$ that satisfy the orthogonality property

$$\sum_{t=a'}^{b'-1} \mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')_q \mathrm{u}_m^{\alpha',\beta'}(x(t), a', b')_q \rho'(t) \Delta x(t - 1/2) = (d_k')^2 \delta_{k,m}, \qquad (49)$$

where $\rho'(t)$ and d_k' are the weight function ρ and the norm d_n given in table 1 with the corresponding change a, b, α, β, s, n by a', b', α', β', t, k.

Furthermore, with the above choice (48) of the parameters of $\mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')_q$, the hypergeometric function ${}_4\mathrm{F}_3$ in (47) coincides with the function ${}_4\mathrm{F}_3$ in (35) and therefore the following relation between the polynomials $\mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')$ and $u_n^{\alpha,\beta}(x(s), a, b)_q$ holds

$$\mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')_q = \mathcal{A}(\alpha, \beta, a, b, n, s) u_n^{\alpha,\beta}(x(s), a, b)_q, \qquad (50)$$

where

$$\mathcal{A}(\alpha, \beta, a, b, n, s) = \frac{(-1)^{s-a+n} \widetilde{\Gamma}_q(b-a-n) \widetilde{\Gamma}_q(s-a+\beta+1) \widetilde{\Gamma}_q(b+\alpha+s+1) \widetilde{\Gamma}_q(n+1)}{\widetilde{\Gamma}_q(b-s) \widetilde{\Gamma}_q(n+\beta+1) \widetilde{\Gamma}_q(b+a+\alpha+n+1) \widetilde{\Gamma}_q(s-a+1)}.$$

If we now substitute (50) in (49) and make the change (48), then (49) becomes into relation (46), i.e., the polynomial set $\mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')_q$ defined by (47) (or (50)) is the dual set associated to the Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$.

To conclude this study, let us show that the TTRR (16) of the polynomials $\mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')_q$ is the SODE (4) of the polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$ whereas the SODE (4) of the $\mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')_q$ becomes into the TTRR (16) of $u_n^{\alpha,\beta}(x(s), a, b)_q$ and vice versa.

Let denote by $\varsigma(t)$ the σ function of the polynomial $\mathrm{u}_k^{\alpha',\beta'}$, then

$$\varsigma(t) = [t-a']_q [t+b']_q [t+a'-\beta']_q [b'+\alpha'-t]_q = [n]_q [n+b-a+\alpha+\beta]_q [n+\alpha]_q [b+a-n-\beta]_q,$$

and therefore,

$$\varsigma(-t-1) = [\alpha+\beta+n+1]_q [b+a+\alpha+n+1]_q [b-a-n-1]_q [n+\beta+1]_q,$$

$$\lambda_k = [k]_q [\alpha'+\beta'+k+1]_q = [s-a]_q [s+a+1]_q.$$

For the coefficients α_k', β_k' and γ_k', of the TTRR for the polynomials $\mathrm{u}_k^{\alpha',\beta'}$ we have

$$\alpha_k' = \frac{[k+1]_q [\alpha'+\beta'+k+1]_q}{[\alpha'+\beta'+2k+1]_q [\alpha'+\beta'+2k+2]_q} = \frac{[s-a+1]_q [s+a+1]_q}{[2s+1]_q [2s+2]_q},$$

$$\gamma_k' = \frac{[b+\alpha+s]_q [b+\alpha-s]_q [s+a-\beta]_q [s-a+\beta]_q [b+s]_q [b-s]_q}{[2s+1]_q [2s]_q},$$

and

$$\beta_k' = [n + \frac{\alpha+\beta}{2}]_q [n + \frac{\alpha+\beta}{2} + 1]_q + \frac{\sigma(-s-1)}{[2s+1]_q [2s+2]_q} + \frac{\sigma(s)}{[2s+1]_q [2s]_q}.$$

Also we have $\Delta x(t) = [2t+2]_q = [2n+\alpha+\beta+2]_q$ and $x(s) = [s]_q [s+1]_q = [k+a]_q [k+a+1]_q$.

Let show that the SODE of the Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$ is the TTRR of the $\mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')_q$ polynomials. First, we substitute the relation (50) in the SODE (4) of the polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$ and use that $u_n^{\alpha,\beta}(x(s \pm 1), a, b)_q$ is proportional to $\mathrm{u}_{k\pm 1}^{\alpha',\beta'}(x(t), a', b')_q$ (see (50)). After some simplification, and using the last formulas we obtain

$$\alpha_k' \mathrm{u}_{k+1}^{\alpha',\beta'}(x(t), a', b')_q + \Big(\beta_k' - [n]_q [\alpha+\beta+n+1]_q - [\tfrac{\alpha+\beta}{2}]_q [\tfrac{\alpha+\beta}{2} + 1]_q\Big) \mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')_q$$
$$+ \gamma_k' \mathrm{u}_{k-1}^{\alpha',\beta'}(x(t), a', b')_q = 0,$$

but

$$[n]_q [\alpha+\beta+n+1]_q + [\tfrac{\alpha+\beta}{2}]_q [\tfrac{\alpha+\beta}{2} + 1]_q = [n + \tfrac{\alpha+\beta}{2}]_q [n + \tfrac{\alpha+\beta}{2} + 1]_q = x(t),$$

i.e., we obtain the TTRR for the polynomials $\mathrm{u}_k^{\alpha',\beta'}(x(t),a',b')_q$.

If we now substitute (50) in the TTRR (16) for the Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$, and use that $u_{n\pm1}^{\alpha,\beta}(x(s),a,b)_q \sim \mathrm{u}_k^{\alpha',\beta'}(x(t\pm1),a',b')_q$, then we obtain the SODE

$$
\begin{aligned}
&\frac{\varsigma(-t-1)}{\Delta x(t)\Delta x(t-\frac12)}\mathrm{u}_k^{\alpha',\beta'}(x(t+1),a',b')_q+\frac{\varsigma(t)}{\nabla x(t)\Delta x(t-\frac12)}\mathrm{u}_k^{\alpha',\beta'}(x(t-1),a',b')_q\\
&-\left[\frac{\varsigma(-t-1)}{\Delta x(t)\Delta x(t-\frac12)}+\frac{\varsigma(t)}{\nabla x(t)\Delta x(t-\frac12)}+[a]_q[a+1]_q-[k+a]_q[k+a+1]_q\right]\mathrm{u}_k^{\alpha',\beta'}(x(t),a',b')_q=0.
\end{aligned}
$$

That is the SODE (4) of the $\mathrm{u}_k^{\alpha',\beta'}(x(t),a',b')_q$ since

$$
[a]_q[a+1]_q-[k+a]_q[k+a+1]_q=-[k]_q[k+2a+1]_q=-[k]_q[k+\alpha'+\beta'+1]_q=-\lambda_k.
$$

2.2 The q-Racah polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$

There is another possibility to define the q-Racah polynomials as it is suggested in [17, 20]. It corresponds to the function

$$
\sigma(s)=[s-a]_q[s+b]_q[s-a+\beta]_q[b+\alpha+s]_q,
$$

i.e., $A=1$, $s_1=a$, $s_2=-b$, $s_3=a-\beta$, $s_4=-b-\alpha$. With this choice we obtain a new family of polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ that is orthogonal with respect to the weight function

$$
\rho(s)=\frac{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s+a-\beta+1)}{\widetilde{\Gamma}_q(s+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(b-s)}.
$$

All their characteristics can be obtained exactly in the same way as before. Moreover, they can be also obtained from the corresponding characteristics of the polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ by changing $\alpha\to-2b-\alpha$, $\beta\to2a-\beta$ —and using the properties of the functions $\widetilde{\Gamma}_q(s)$, $\Gamma_q(s)$, $(a|q)_n$ and $(a;q)_n$. We will resume the main data of the polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ in table 2.

2.2.1 The hypergeometric representation

For the $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ polynomials we have the following hypergeometric representation

$$
\begin{aligned}
\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q=&\frac{q^{-\frac{n}{2}(4a-2b-\alpha-\beta+n+1)}(q^{a-b+1};q)_n(q^{2a-\beta+1};q)_n(q^{a-b-\alpha+1};q)_n}{\varkappa_q^{2n}(q;q)_n}\times\\
&{}_4\varphi_3\left(\begin{matrix}q^{-n},q^{2a-2b-\alpha-\beta+n+1},q^{a-s},q^{a+s+1}\\ q^{a-b+1},q^{2a-\beta+1},q^{a-b-\alpha+1}\end{matrix}\,\middle|\,q\,,\,q\right),
\end{aligned}\tag{51}
$$

or, in terms of the q-hypergeometric series (28),

$$
\begin{aligned}
\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q=&\frac{(a-b+1|q)_n(2a-\beta+1|q)_n(a-b-\alpha+1|q)_n}{[n]_q!}\times\\
&{}_4\mathrm{F}_3\left(\begin{matrix}-n,2a-2b-\alpha-\beta+n+1,a-s,a+s+1\\ a-b+1,2a-\beta+1,a-b-\alpha+1\end{matrix}\,\middle|\,q\,,\,1\right).
\end{aligned}\tag{52}
$$

Using the Sears transformation formula [11, Eq. (III.15)] we obtain the equivalent representation formulas

$$
\begin{aligned}
\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q=&\frac{q^{-\frac{n}{2}(2a-4b-\alpha-\beta+n+1)}(q^{a-b+1};q)_n(q^{-2b-\alpha+1};q)_n(q^{-\beta+a-b+1};q)_n}{\varkappa_q^{2n}(q;q)_n}\times\\
&{}_4\varphi_3\left(\begin{matrix}q^{-n},q^{2a-2b-\alpha-\beta+n+1},q^{-b-s},q^{-b+s+1}\\ q^{a-b+1},q^{-2b-\alpha+1},q^{a-b-\beta+1}\end{matrix}\,\middle|\,q\,,\,q\right),
\end{aligned}\tag{53}
$$

and

$$
\begin{aligned}
\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q=&\frac{(a-b+1|q)_n(-2b-\alpha+1|q)_n(a-b-\beta+1|q)_n}{[n]_q!}\times\\
&{}_4\mathrm{F}_3\left(\begin{matrix}-n,2a-2b-\alpha-\beta+n+1,-b-s,-b+s+1\\ a-b+1,-2b-\alpha+1,a-b-\beta+1\end{matrix}\,\middle|\,q\,,\,1\right).
\end{aligned}\tag{54}
$$

Table 2: Main data of the q-Racah polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$

$P_n(s)$	$\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q\,,\quad x(s)=[s]_q[s+1]_q$
(a,b)	$[a,b-1]$
$\rho(s)$	$\dfrac{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s+a-\beta+1)}{\widetilde{\Gamma}_q(s+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(b-s)}$ $-\frac{1}{2}<a\le b-1, \alpha>-1, -1<\beta<2a+1$
$\sigma(s)$	$[s-a]_q[s+b]_q[s-a+\beta]_q[b+\alpha+s]_q$
$\sigma(-s-1)$	$[s+a+1]_q[b-s-1]_q[s+a-\beta+1]_q[b+\alpha-s-1]_q$
$\tau(s)$	$[2a-\beta+1]_q[b]_q[b+\alpha]_q-[2b+\alpha-1]_q[a]_q[a-\beta]_q-[2b+\alpha-1]_q[2a-\beta+1]_q$ $-[2b-2a+\alpha+\beta-2]_q x(s)$
$\tau_n(s)$	$-[2b-2a+\alpha+\beta-2n-2]_q x(s+\frac{n}{2})+[a+\frac{n}{2}+1]_q[b-\frac{n}{2}-1]_q[a+\frac{n}{2}+1-\beta]_q[b-\frac{n}{2}+\alpha-1]_q$ $-[a+\frac{n}{2}]_q[b-\frac{n}{2}]_q[a+\frac{n}{2}-\beta]_q[b-\frac{n}{2}+\alpha]_q$
λ_n	$[n]_q[2b-2a+\alpha+\beta-n-1]_q$
B_n	$\dfrac{1}{[n]_q!}$
d_n^2	$\dfrac{\widetilde{\Gamma}_q(2a+n-\beta+1)\widetilde{\Gamma}_q(2b-2a+\alpha+\beta-n)[2b-2a-2n-1+\alpha+\beta]_q^{-1}}{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(b-a-n+\alpha)\widetilde{\Gamma}_q(b-a+\beta-n)\widetilde{\Gamma}_q(2b+\alpha-n)\widetilde{\Gamma}_q(b-a+\alpha+\beta-n)}$
$\rho_n(s)$	$\dfrac{\widetilde{\Gamma}_q(s+a+n+1)\widetilde{\Gamma}_q(s+a+n-\beta+1)}{\widetilde{\Gamma}_q(s+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s-n)\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(b-s-n)}$
a_n	$\dfrac{(-1)^n\widetilde{\Gamma}_q[2b-2a+\alpha+\beta-n]_q}{[n]_q!\widetilde{\Gamma}_q[2b-2a+\alpha+\beta-2n]_q}$
α_n	$-\dfrac{[n+1]_q[2b-2a+\alpha+\beta-n-1]_q}{[2b-2a+\alpha+\beta-2n-1]_q[2b-2a+\alpha+\beta-2n-2]_q}$
β_n	$[a]_q[a+1]_q+\dfrac{[2b-2a+\alpha+\beta-n-1]_q[a-b+n+1]_q[2a-\beta+n+1]_q[a-b-\alpha+n+1]_q}{[2b-2a+\alpha+\beta-2n-1]_q[2b-2a+\alpha+\beta-2n-2]_q}$ $+\dfrac{[2b+\alpha-n]_q[b-a+\alpha+\beta-n]_q[b-a+\beta-n]_q[n]_q}{[2b-2a+\alpha+\beta-2n-1]_q[2b-2a+\alpha+\beta-2n]_q}$
γ_n	$-\dfrac{[2a-\beta+n]_q[b-a-n]_q[b-a-n+\alpha]_q[b-a-n+\beta]_q[2b+\alpha-n]_q[b-a+\alpha+\beta-n]_q}{[2b-2a+\alpha+\beta-2n-1]_q[2b-2a+\alpha+\beta-2n]_q}$

Remark: From the above formulas follow that the polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ are multiples of the standard q-Racah polynomials $R_n(\mu(q^{a-s});q^{a-b-\alpha},q^{a-b-\beta},q^{a-b},q^{a+b}|q)$.

Moreover, from the above hypergeometric representations follow the values

$$\begin{aligned}\widetilde{u}_n^{\alpha,\beta}(x(a),a,b)_q =&\frac{(a-b+1|q)_n(2a-\beta+1|q)_n(a-b-\alpha+1|q)_n}{[n]_q!}\\ =&\frac{(q^{a-b+1};q)_n(q^{2a-\beta+1};q)_n(q^{a-b-\alpha+1};q)_n}{q^{\frac{n}{2}(4a-2b-\alpha-\beta+n+1)}\varkappa_q^{2n}(q;q)_n},\end{aligned}\tag{55}$$

$$\begin{aligned}\widetilde{u}_n^{\alpha,\beta}(x(b-1),a,b)_q =&\frac{(a-b+1|q)_n(-2b-\alpha+1|q)_n(a-b-\beta+1|q)_n}{[n]_q!}\\ =&\frac{(q^{a-b+1};q)_n(q^{-2b-\alpha+1};q)_n(q^{-\beta+a-b+1};q)_n}{q^{\frac{n}{2}(2a-4b-\alpha-\beta+n+1)}\varkappa_q^{2n}(q;q)_n}.\end{aligned}\tag{56}$$

Using (10) we obtain an explicit formula[3]

$$\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q = \frac{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(b-s)\widetilde{\Gamma}_q(s+\alpha+b+1)}{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s+a-\beta+1)}\times$$

$$\widetilde{\Gamma}_q(b+\alpha-s)\sum_{k=0}^{n}\frac{(-1)^{k+n}[2s+2k-n+1]_q\widetilde{\Gamma}_q(s+k+a+1)\widetilde{\Gamma}_q(2s+k-n+1)}{\widetilde{\Gamma}_q(k+1)\widetilde{\Gamma}_q(n-k+1)\widetilde{\Gamma}_q(2s+k+2)\widetilde{\Gamma}_q(s-n+k-a+1)\widetilde{\Gamma}_q(b-s-k)}\times \tag{57}$$

$$\frac{\widetilde{\Gamma}_q(s+k+a-\beta+1)}{\widetilde{\Gamma}_q(s+k-n+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s-k)\widetilde{\Gamma}_q(s-n+k+b+1)\widetilde{\Gamma}_q(s-n+k-a+\beta+1)}.$$

From this expression follows that

$$\begin{aligned}\widetilde{u}_n^{\alpha,\beta}(x(a),a,b)_q &= \frac{\widetilde{\Gamma}_q(b-a)\widetilde{\Gamma}_q(2a-\beta+n+1)\widetilde{\Gamma}_q(b-a+\alpha)}{[n]!\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(2a-\beta+1)\widetilde{\Gamma}_q(b-a+\alpha-n)},\\ \widetilde{u}_n^{\alpha,\beta}(x(b-1),a,b)_q &= \frac{(-1)^n\widetilde{\Gamma}_q(b-a)\widetilde{\Gamma}_q(2b+\alpha)\widetilde{\Gamma}_q(b-a+\beta)}{[n]!\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(2b+\alpha-n)\widetilde{\Gamma}_q(b-a+\beta-n)},\end{aligned} \tag{58}$$

that are in agreement with the values (55) and (56) obtained before.

From the hypergeometric representation follows the symmetry property

$$\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q = \widetilde{u}_n^{-b-a+\beta,b+a+\alpha}(x(s),a,b)_q.$$

2.2.2 The differentiation formulas

Next we use the differentiation formulas (21) and (23) to obtain

$$\frac{\Delta\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q}{\Delta x(s)} = -[2b-2a+\alpha+\beta-n-1]_q\widetilde{u}_{n-1}^{\alpha,\beta}(x(s+\tfrac{1}{2}),a+\tfrac{1}{2},b-\tfrac{1}{2})_q, \tag{59}$$

$$\begin{aligned}[][n]_q[2s+1]_q\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q =& \sigma(-s-1)\widetilde{u}_{n-1}^{\alpha,\beta}(x(s+\tfrac{1}{2}),a+\tfrac{1}{2},b-\tfrac{1}{2})_q\\ &-\sigma(s)\widetilde{u}_{n-1}^{\alpha,\beta}(x(s-\tfrac{1}{2}),a+\tfrac{1}{2},b-\tfrac{1}{2})_q,\end{aligned} \tag{60}$$

respectively. Finally, the formulas (24) (or (25)) and (26) lead to the following *differentiation formulas*

$$\sigma(s)\frac{\nabla\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q}{[2s]_q} = -\frac{[2b-2a+\alpha+\beta-n-1]_q}{[2b-2a+\alpha+\beta-2n-2]_q}\Big[\tau_n(s)\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q-[n+1]_q\widetilde{u}_{n+1}^{\alpha,\beta}(x(s),a,b)_q\Big], \tag{61}$$

$$\sigma(-s-1)\frac{\Delta\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q}{[2s+2]_q} = -\frac{[2b-2a+\alpha+\beta-n-1]_q}{[2b-2a+\alpha+\beta-2n-2]_q}\times$$

$$\Big[(\tau_n(s)+[n]_q[2b-2a+\alpha+\beta-2n-2]_q[2s+1]_q)\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q-[n+1]_q\widetilde{u}_{n+1}^{\alpha,\beta}(x(s),a,b)_q\Big], \tag{62}$$

respectively, where $\tau_n(s)$ is given in table 2.

2.3 The dual set to $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$

To obtain the dual set to $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ we use the same method as in the previous section. We start from the orthogonality relation (13) for the $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ polynomials defined by (54) and write the dual relation

$$\sum_{n=0}^{N-1}\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q\widetilde{u}_n^{\alpha,\beta}(x(s'),a,b)_q\frac{1}{d_n^2} = \frac{1}{\rho(s)\Delta x(s-1/2)}\delta_{s,s'},\quad N=b-a, \tag{63}$$

[3]Obviously the formulas (51–54) also give two equivalent explicit formulas.

where ρ and d_n^2 are the weight function and the norm of the $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ given in table 2. Furthermore, from (54) follows that the functions $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ are polynomials of degree $k = b-s-1$ on the lattice $x(t) = [t]_q[t+1]_q$ where $t = b-a-n+\frac{\alpha+\beta}{2}-1$ (the proof is similar to the one presented in section 2.1.4 and we will omit it here). To identify the dual set let us define the new set

$$\begin{aligned}\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q =& \frac{(-1)^k\widetilde{\Gamma}_q(b'-a')\widetilde{\Gamma}_q(b'-a'+\beta')\widetilde{\Gamma}_q(2b'+\alpha')}{[k]!\widetilde{\Gamma}_q(b'-a'-k)\widetilde{\Gamma}_q(b'-a'+\beta'-k)\widetilde{\Gamma}_q(2b'+\alpha'-k)}\times \\ & {}_4\mathrm{F}_3\left(\begin{array}{c} -k, 2a'-2b'-\alpha'-\beta'+k+1, -b'-t, -b'+t+1 \\ a'-b'+1, -2b'-\alpha'+1, a'-b'-\beta'+1 \end{array}\,\middle|\, q\,,\,1\right),\end{aligned} \tag{64}$$

where

$$k = b-s-1, \quad t = b-a-n+\frac{\alpha+\beta}{2}-1, \quad a' = \frac{\alpha+\beta}{2}, \quad b' = b-a+\frac{\alpha+\beta}{2}, \quad \alpha' = 2a-\beta, \quad \beta' = \beta. \tag{65}$$

Obviously they satisfy the following orthogonality relation

$$\sum_{t=a'}^{b'-1} \widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q \widetilde{\mathrm{u}}_m^{\alpha',\beta'}(x(t),a',b')_q \rho'(t)\Delta x(t-1/2) = (d_k')^2\delta_{k,m}, \tag{66}$$

where now $\rho'(t)$ and d_k' are the weight function ρ and the norm d_n, respectively, given in table 2 with the corresponding change of the parameters $a, b, \alpha, \beta, n, s$ by $a', b', \alpha', \beta', k, t$ (65).

Furthermore, with the above definition (65) for the parameters of $\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q$, the hypergeometric function ${}_4\mathrm{F}_3$ in (64) coincides with the function ${}_4\mathrm{F}_3$ in (54) and therefore the following relation between the polynomials $\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')$ and $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ holds

$$\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q = \widetilde{\mathcal{A}}(\alpha,\beta,a,b,n,s)\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q, \tag{67}$$

where

$$\widetilde{\mathcal{A}}(\alpha,\beta,a,b,n,s) = \frac{(-1)^{b-s-1-n}\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(2b+\alpha-n)\widetilde{\Gamma}_q(b-a+\beta-n)\widetilde{\Gamma}_q(n+1)}{\widetilde{\Gamma}_q(b-s)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(s+b+\alpha+1)\widetilde{\Gamma}_q(s-a+1)}.$$

To prove that the polynomials $\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q$ are the dual set to $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ it is sufficient to substitute (67) in (66) and do the change (65) that transforms (66) into (46).

Let also mention that, as in the case of the q-Racah polynomials, the TTRR (16) of the polynomials $\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q$ is the SODE (4) of the polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ whereas the SODE (4) of the $\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q$ becomes into the TTRR (16) of $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ and vice versa.

To conclude this section let us point out that there exist a simple relation connecting both polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ and $\tilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ (see (87) from below). We will establish it at the end of the next section.

3 Connection with the $6j$-symbols of the q-algebra $SU_q(2)$

3.1 $6j$-symbols of the quantum algebra $SU_q(2)$

It is known (see e.g. [26] and references therein) that the Racah coefficients $U_q(j_1\,j_2\,j\,j_3; j_{12}\,j_{23})$ are used for the transition from the coupling scheme of three angular momenta j_1, j_2, j_3

$$|j_1j_2(j_{12}),j_3:jm\rangle = \sum_{m_1,m_2,m_3,m_{12}} \langle j_1m_1j_2m_2|j_{12}m_{12}\rangle\langle j_{12}m_{12}j_3m_3|jm\rangle|j_1m_1\rangle|j_2m_2\rangle|j_3m_3\rangle,$$

to the following ones

$$|j_1j_2j_3(j_{23}):jm\rangle = \sum_{m_1,m_2,m_3,m_{23}} \langle j_2m_2j_3m_3|j_{23}m_{23}\rangle\langle j_1m_1j_{23}m_{23}|jm\rangle|j_1m_1\rangle|j_2m_2\rangle|j_3m_3\rangle,$$

where $\langle j_a m_a j_b m_b | j_{ab} m_{ab}\rangle$ denotes the Clebsh-Gordon Coefficients of the quantum algebra $su_q(2)$. In fact we have that the recoupling is given by

$$|j_1 j_2(j_{12}), j_3 : jm\rangle = \sum_{j_{23}} U_q(j_1\, j_2\, j\, j_3; j_{12}\, j_{23}) |j_1 j_2 j_3(j_{23}) : jm\rangle.$$

The Racah coefficients U define an unitary matrix, i.e., they satisfy the orthogonality relations

$$\sum_{j_{23}} U_q(j_1\, j_2\, j\, j_3; j_{12}\, j_{23}) U_q(j_1\, j_2\, j\, j_3; j'_{12}\, j_{23}) = \delta_{j_{12}, j'_{12}}, \tag{68}$$

$$\sum_{j_{12}} U_q(j_1\, j_2\, j\, j_3; j_{12}\, j_{23}) U_q(j_1\, j_2\, j\, j_3; j_{12}\, j'_{23}) = \delta_{j_{23}, j'_{23}}. \tag{69}$$

Usually instead of the Racah coefficients is more convenient to use the $6j$-symbols defined by

$$U_q(j_1\, j_2\, j\, j_3; j_{12}\, j_{23}) = (-1)^{j_1+j_2+j_3+j} \sqrt{[2j_{12}+1]_q [2j_{23}+1]_q} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q .$$

The $6j$-symbols have the following symmetry property

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = \left\{ \begin{array}{ccc} j_3 & j_2 & j_{23} \\ j_1 & j & j_{12} \end{array} \right\}_q . \tag{70}$$

Here and without lost of generality we will suppose that $j_1 \geq j_2$ and $j_3 \geq j_2$, then for the moments j_{23} and j_{12} we have the intervals

$$j_3 - j_2 \leq j_{23} \leq j_2 + j_3, \quad j_1 - j_2 \leq j_{12} \leq j_1 + j_2,$$

respectively. Now, in order to avoid any other restrictions on these two momenta (caused by the so called triangle inequalities for the $6j$-symbols) we will assume that the following restrictions hold

$$|j - j_3| \leq \min(j_{12}) = j_1 - j_2, \quad |j - j_1| \leq \min(j_{23}) = j_3 - j_2.$$

3.2 $6j$-symbols and the q-Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$

Now we are ready to establish the connection of $6j$-symbols with the q-Racah polynomials. We fix the variable s as $s = j_{23}$ that runs on the interval $a \leq s \leq b-1$ where $a = j_3 - j_2$, $b = j_2 + j_3 + 1$. Let us put

$$(-1)^{j_1+j_{23}+j} \sqrt{[2j_{12}+1]_q} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = \sqrt{\frac{\rho(s)}{d_n^2}}\, u_n^{\alpha,\beta}(x(s), a, b)_q, \tag{71}$$

where $\rho(s)$ and d_n are the weight function and the norm, respectively, of the q-Racah polynomials on the lattice (1) $u_n^{\alpha,\beta}(x(s), a, b)_q$, and $n = j_{12} - j_1 + j_2$, $\alpha = j_1 - j_2 - j_3 + j \geq 0$, $\beta = j_1 - j_2 + j_3 - j \geq 0$.[4]

To verify the above relation we use the recurrence relation [27, Eq. (5.17)]

$$\begin{aligned} & [2]_q [2j_{23}+2]_q A_q^- \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}-1 \end{array} \right\}_q - \\ & \Big(([2j_{23}]_q [2j_1+2]_q - [2]_q [j - j_{23} + j_1 + 1]_q [j + j_{23} - j_1]_q) \times \\ & ([2j_2]_q [2j_{23}+2]_q - [2]_q [j_3 - j_2 + j_{23} + 1]_q [j_3 + j_2 - j_{23}]_q) - \\ & ([2j_2]_q [2j_1+2]_q - [2]_q [j_{12} - j_2 + j_1 + 1]_q [j_{12} + j_2 - j_1]_q) [2j_{23}+2]_q [2j_{23}]_q \Big) \times \\ & [2j_{23}+1]_q \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q + [2]_q [2j_{23}]_q A_q^+ \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}+1 \end{array} \right\}_q = 0, \end{aligned} \tag{72}$$

[4]Notice that this is equivalent to the following setting

$$j_1 = (b-a-1+\alpha+\beta)/2, \quad j_2 = (b-a-1)/2, \quad j_3 = (a+b-1)/2,$$
$$j_{12} = (2n+\alpha+\beta)/2, \quad j_{23} = s, \quad j = (a+b-1+\alpha-\beta)/2.$$

where

$$
\begin{aligned}
A_q^- =& \sqrt{[j+j_{23}+j_1+1]_q[j+j_{23}-j_1]_q[j-j_{23}+j_1+1]_q[j_{23}-j+j_1]_q}\times \\
& \sqrt{[j_2+j_3+j_{23}+1]_q[j_2+j_3-j_{23}+1]_q[j_3-j_2+j_{23}]_q[j_2-j_3+j_{23}]_q}, \\
A_q^+ =& \sqrt{[j+j_{23}+j_1+2]_q[j+j_{23}-j_1+1]_q[j-j_{23}+j_1]_q[j_{23}-j+j_1+1]_q}\times \\
& \sqrt{[j_2+j_3+j_{23}+2]_q[j_2+j_3-j_{23}]_q[j_3-j_2+j_{23}+1]_q[j_2-j_3+j_{23}+1]_q}.
\end{aligned}
\tag{73}
$$

Notice that

$$
A_q^- = \sqrt{\sigma(j_{23})\sigma(-j_{23})}, \quad A_q^+ = \sqrt{\sigma(j_{23}+1)\sigma(-j_{23}-1)},
$$

where

$$
\begin{aligned}
\sigma(j_{23}) &= [j_{23}-j_3+j_2]_q[j_{23}+j_2+j_3+1]_q[j_{23}-j_1+j]_q[j+j_1-j_{23}+1]_q, \\
\sigma(-j_{23}-1) &= [j_{23}+j_3-j_2+1]_q[j_2+j_3-j_{23}]_q[j_{23}+j_1-j+1]_q[j+j_1+j_{23}+2]_q.
\end{aligned}
$$

Substituting (71) in (72) and simplifying the obtained expression we get

$$
\begin{aligned}
&[2s]_q\sigma(-s-1)u_n^{\alpha,\beta}(x(s+1),a,b)_q + [2s+2]_q\sigma(s)u_n^{\alpha,\beta}(x(s-1),a,b)_q + \\
&\Big(\lambda_n[2s]_q[2s+1]_q[2s+2]_q - [2s]_q\sigma(-s-1) - [2s+2]_q\sigma(s)\Big)u_n^{\alpha,\beta}(x(s),a,b)_q = 0,
\end{aligned}
$$

which is the difference equation for the q-Racah polynomials (4). Since $u_0^{\alpha,\beta}(x(s),a,b)_q = 1$, (71) leads to

$$
\begin{aligned}
&(-1)^{j_1+j_{23}+j}\sqrt{[2j_1-2j_2+1]_q}\left\{\begin{array}{ccc} j_1 & j_2 & j_1-j_2 \\ j_3 & j & j_{23} \end{array}\right\}_q = \sqrt{\frac{\rho(s)}{d_0^2}} \quad \Rightarrow \\
&\left\{\begin{array}{ccc} j_1 & j_2 & j_1-j_2 \\ j_3 & j & j_{23} \end{array}\right\}_q := \left\{\begin{array}{ccc} j_1 & j_2 & j_1-j_2 \\ j_3 & j & s \end{array}\right\}_q \\
&=(-1)^{j+j_1+s}\sqrt{\frac{[j_1+j+s+1]_q![j_1+j-s]_q![j_1-j+s]_q![j_3-j_2+s]_q!}{[j-j_1+s]_q![j_3+j_2-s]_q![j_2-j_3+s]_q![j_2+j_3+s+1]_q!}}\times \\
&\sqrt{\frac{[2j_1-2j_2]_q![2j_2]_q![j_2+j_3+j-j_1]_q!}{[2j_1+1]_q![j_1+j_3-j_2-j]_q![j_1-j_3-j_2+j]_q![j_1+j_3-j_2+j+1]_q!}}.
\end{aligned}
$$

Furthermore, substituting the values $s = a$ and $s = b-1$ in (71) and using (41) we find

$$
\begin{aligned}
&\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_3-j_2 \end{array}\right\}_q = (-1)^{j_{12}+j_3+j}\times \\
&\sqrt{\frac{[j_{12}+j_3-j]_q![2j_2]_q![j_{12}+j_3+j+1]_q![2j_3-2j_2]_q![j_2-j_1+j_{12}]_q!}{[j_1-j_2+j_3-j]_q![j_1+j_2-j_{12}]_q![j_1-j_2+j_3+j+1]_q!}}\times \\
&\sqrt{\frac{[j_1+j_2-j_3+j]_q![j_1-j_2+j_{12}]_q![j_3-j_{12}+j]_q!}{[2j_3+1]_q![j_3-j_1-j_2+j]_q![j_{12}-j_3+j]_q![j_1+j_2+j_{12}+1]_q!}}
\end{aligned}
\tag{74}
$$

and

$$
\begin{aligned}
&\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_2+j_3 \end{array}\right\}_q = (-1)^{j_1+j_2+j_3+j}\times \\
&\sqrt{\frac{[2j_2]_q![j_{12}-j_3+j]_q![j_2-j_1+j_3+j]_q![2j_3]_q![j_1+j_2+j_3-j]_q!}{[j_1+j_2-j_{12}]_q![j_1-j_2-j_3+j]_q![j_3-j_{12}+j]_q!}}\times \\
&\sqrt{\frac{[j_2-j_1+j_{12}]_q![j_1-j_2+j_{12}]_q![j_1+j_2+j_3+j+1]_q!}{[2j_2+2j_3+1]_q![j_{12}+j_3-j]_q![j_1+j_2+j_{12}+1]_q![j_{12}+j_3+j+1]_q!}},
\end{aligned}
\tag{75}
$$

that are in agreement with the results in [26].

The relation (71) allows us to obtain several recurrence relations for the $6j$-symbols of the quantum algebra $SU_q(2)$ by using the properties of the q-Racah polynomials. So, the TTRR (16) gives

$$\begin{aligned}
&[2j_{12}]_q\,\widetilde{A}_q^{+}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}+1\\ j_3 & j & j_{23}\end{array}\right\}_q+[2j_{12}+2]_q\,\widetilde{A}_q^{-}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}-1\\ j_3 & j & j_{23}\end{array}\right\}_q\\
&-\Big([2j_{12}]_q[2j_{12}+1]_q[2j_{12}+2]_q\big([j_{23}]_q[j_{23}+1]_q-[j_3-j_2]_q[j_3-j_2+1]_q\big)+[2j_{12}]_q\times\\
&[j_1-j_2+j_{12}+1]_q[j_{12}-j_1-j_2]_q[j_{12}+j_3-j+1]_q[j_{12}+j_3+j+2]_q-[2j_{12}+2]_q\times\\
&[j_{12}-j_3+j]_q[j_1+j_2+j_{12}+1]_q[j_3-j_{12}+j+1]_q[j_2-j_1+j_{12}]_q\Big)\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}\end{array}\right\}_q=0,
\end{aligned}\tag{76}$$

where

$$\begin{aligned}
\widetilde{A}_q^{-}=&\;\sqrt{[j_2-j_1+j_{12}]_q[j_1-j_2+j_{12}]_q[j_{12}-j_3+j]_q[j_{12}+j_3-j]_q[j_1+j_2+j_{12}+1]_q}\times\\
&\sqrt{[j_{12}+j_3+j+1]_q[j_1+j_2-j_{12}+1]_q[j_3-j_{12}+j+1]_q}\\
\widetilde{A}_q^{+}=&\;\sqrt{[j_2-j_1+j_{12}+1]_q[j_1-j_2+j_{12}+1]_q[j_{12}-j_3+j+1]_q[j_{12}+j_3-j+1]_q}\times\\
&\sqrt{[j_1+j_2+j_{12}+2]_q[j_{12}+j_3+j+2]_q[j_1+j_2-j_{12}]_q[j_3-j_{12}+j]_q}.
\end{aligned}\tag{77}$$

The expressions (42) and (43) yield

$$\begin{aligned}
&\sqrt{\sigma(j_{23}+1)}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}+1\end{array}\right\}_q+\sqrt{\sigma(-j_{23}-1)}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}\end{array}\right\}_q\\
&=[2j_{23}+2]_q\sqrt{[j_2-j_1+j_{12}]_q[j_1-j_2+j_{12}+1]_q}\left\{\begin{array}{ccc} j_1+\frac12 & j_2-\frac12 & j_{12}\\ j_3 & j & j_{23}+\frac12\end{array}\right\}_q,
\end{aligned}\tag{78}$$

and

$$\begin{aligned}
&\sqrt{\sigma(-j_{23}-1)}\left\{\begin{array}{ccc} j_1+\frac12 & j_2-\frac12 & j_{12}\\ j_3 & j & j_{23}+\frac12\end{array}\right\}_q+\sqrt{\sigma(j_{23})}\left\{\begin{array}{ccc} j_1+\frac12 & j_2-\frac12 & j_{12}\\ j_3 & j & j_{23}-\frac12\end{array}\right\}_q\\
&=[2j_{23}+1]_q\sqrt{[j_{12}-j_1+j_2]_q[j_{12}+j_1-j_2+1]_q}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}\end{array}\right\}_q,
\end{aligned}\tag{79}$$

respectively, whereas the differentiation formulas (44)–(45) give

$$\begin{aligned}
&[2j_{12}+2]_qA_q^{-}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}-1\end{array}\right\}_q+[2j_{23}]_q\widetilde{A}_q^{+}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}+1\\ j_3 & j & j_{23}\end{array}\right\}_q+\\
&\Big(\sigma(j_{23})[2j_{12}+2]_q+[j_1-j_2+j_{12}+1]_q[2j_{23}]_q\Lambda(j_{12},j_{23},j_1,j_2)\Big)\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}\end{array}\right\}_q=0
\end{aligned}\tag{80}$$

and

$$\begin{aligned}
&[2j_{12}+2]_qA_q^{+}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}+1\end{array}\right\}_q-[2j_{23}+2]_q\widetilde{A}_q^{+}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}+1\\ j_3 & j & j_{23}\end{array}\right\}_q+\\
&\Big([2j_{12}+2]_q\sigma(-j_{23}-1)-[2j_{23}+2]_q[j_1-j_2+j_{12}+1]_q\,(\Lambda(j_{12},j_{23},j_1,j_2)+\\
&\quad[j_{12}-j_1+j_2]_q[2j_{12}+2]_q[2j_{23}+1]_q)\Big)\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}\end{array}\right\}_q=0,
\end{aligned}\tag{81}$$

respectively, where $A_q^{\pm}$ are given by (73), $\widetilde{A}_q^{\pm}$ by (77) and

$$\begin{aligned}
\Lambda(j_{12},j_{23},j_1,j_2)=&\,\sigma\left(\frac{-j_{12}+j_1-j_2}{2}-1\right)-\sigma\left(\frac{-j_{12}+j_1-j_2}{2}\right)-\\
&[2j_{12}+2]_q\left[j_{23}+\frac{j_{12}-j_1+j_2}{2}\right]_q\left[j_{23}+\frac{j_{12}-j_1+j_2}{2}+1\right]_q.
\end{aligned}$$

Using the hypergeometric representations (35) and (37) we obtain the representation of the $6j$-symbols in terms of the q-hypergeometric function[5] (28)

$$\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array}\right\}_q = (-1)^{j_{12}+j_{23}+j_2+j} \frac{[2j_2]_q!}{[j_1-j_2+j_3-j]_q![j_1-j_2+j_3+j+1]_q!}$$
$$\sqrt{\frac{[j_1+j+j_{23}+1]_q![j_1+j-j_{23}]_q![j_1-j+j_{23}]_q![j_3-j_2+j_{23}]_q!}{[j-j_1+j_{23}]_q![j_3+j_2-j_{23}]_q![j_2-j_3+j_{23}]_q![j_2+j_3+j_{23}+1]_q!}}\times$$
$$\sqrt{\frac{[j_{12}-j_1+j_2]_q![j_{12}+j_1-j_2]_q![j_3+j-j_{12}]_q![j_3+j_{12}-j]_q![j_3+j_{12}+j+1]_q!}{[j_{12}-j_3+j]_q![j_1+j_2+j_{12}+1]_q![j_1+j_2-j_{12}]_q!}}\times$$
$${}_4\mathrm{F}_3\left(\begin{array}{c} j_1-j_2-j_{12}, j_1-j_2+j_{12}+1, j_3-j_2-j_{23}, j_{23}+j_3-j_2+1 \\ -2j_2,\ j_1-j_2+j_3-j+1,\ j_1-j_2+j_3+j+2 \end{array}\,\middle|\, q\,,\,1\right),$$

and

$$\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array}\right\}_q = (-1)^{j_1+j_{23}+j} \frac{[2j_2]_q![j_2+j_3-j_1+j]_q!}{[j_1-j_2-j_3+j]_q!}$$
$$\sqrt{\frac{[j_1+j+j_{23}+1]_q![j_1+j-j_{23}]_q![j_1-j+j_{23}]_q![j_3-j_2+j_{23}]_q!}{[j-j_1+j_{23}]_q![j_3+j_2-j_{23}]_q![j_2-j_3+j_{23}]_q![j_2+j_3+j_{23}+1]_q!}}\times$$
$$\sqrt{\frac{[j_{12}-j_1+j_2]_q![j_{12}+j_1-j_2]_q![j_{12}-j_3+j]_q!}{[j_1+j_2+j_{12}+1]_q![j_1+j_2-j_{12}]_q![j_3+j-j_{12}]_q![j_{12}+j_3-j]_q![j_3+j_{12}+j+1]_q!}}\times$$
$${}_4\mathrm{F}_3\left(\begin{array}{c} j_1-j_2-j_{12}, j_1-j_2+j_{12}+1, -j_3-j_2+j_{23}, -j_{23}-j_3-j_2-1 \\ -2j_2,\ j_1-j_2-j_3+j+1,\ j_1-j_2-j_3-j \end{array}\,\middle|\, q\,,\,1\right).$$

Notice that from the above representations the values (74) and (75) immediately follows. Notice also that the above formulas give two alternative explicit formulas for computing the $6j$-symbols. A third explicit formula follows from (40)

$$\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array}\right\}_q = \sqrt{\frac{[j_{23}+j_2-j_3]_q![j_{23}+j_2+j_3+1]_q![j_{23}+j-j_1]_q![j_2+j_3-j_{23}]_q!}{[j_{23}+j_3-j_2]_q![j_{23}+j_1-j]_q![j_{23}+j_1+j+1]_q![j_1+j-j_{23}]_q!}}\times$$
$$\sqrt{\frac{[j_{12}-j_1+j_2]_q![j_1-j_2+j_{12}]_q![j_1+j_2-j_{12}]_q![j_3-j_{12}+j]_q!}{[j_{12}-j_3+j]_q![j_{12}+j_3-j]_q![j_1+j_2+j_{12}+1]_q![j_{12}+j_3+j]_q!}}\times$$
$$\sum_{k=0}^{j_{12}-j_1+j_2} \frac{(-1)^{k+j_1+j_{23}+j}[2k+j_1-j_2-j_{12}+2j_{23}+1]_q[k+j_{23}+j_3-j_2]_q!}{[k]_q![j_{12}-j_1+j_2-k]_q![2j_3+1+k]_q![k+j_{23}+j_1-j_{12}-j_3]_q!}\times$$
$$\frac{[2j_{23}+k-j_{12}+j_1-j_2]_q![k+j_{23}+j_1-j]_q![k+j_{23}+j_1+j+1]_q![j_1+j-j_{23}-k]_q!}{[k+j_{23}+j_1-j_{12}+j_3+1]_q![k+j_{23}+j-j_2-j_{12}]_q![j_2+j_3-j_{23}+1-k]_q!}$$

To conclude this section let us point out that the orthogonality relations (68) and (69) lead to the orthogonality relations for the Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ (35) and their *duals* $\mathrm{u}_k^{\alpha',\beta'}(x(t),a',b')_q$, respectively, and also that the relation (50) between q-Racah and dual q-Racah corresponds to the symmetry property (70).

3.3 $6j$-symbols and the alternative q-Racah polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$

In this section we will provide the same comparative analysis but for the alternative q-Racah polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$. We again choose $s=j_{23}$ that runs on the interval $[a,b-1]$, $a=j_3-j_2$, $b=j_2+j_3+1$. In this case the connection is given by formula

$$(-1)^{j_{12}+j_3+j}\sqrt{[2j_{12}+1]_q}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array}\right\}_q = \sqrt{\frac{\rho(s)}{d_n^2}}\,\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q, \qquad (82)$$

where $\rho(s)$ and d_n are the weight function and the norm, respectively, of the alternative q-Racah polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ (see Section 2.2) on the lattice (1), and $n=j_1+j_2-j_{12}$, $\alpha=j_1-j_2-j_3+j\ge 0$, $\beta=j_1-j_2+j_3-j\ge 0$.

[5] To obtain the representation in terms of the basic hypergeometric series it is sufficient to use the relation (29).

Using the above relations we see that the SODE (4) for the $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ polynomials becomes into the recurrence relation (72) as well as the TTRR (16) becomes into the recurrence relation (76). Evaluating (82) in $s=j_{23}=j_3-j_2$ and $s=j_{23}=j_2+j_3+1$ and using (58) we recover the values (74) and (75), respectively. If we now put $n=0$, i.e., $j_{12}=j_1+j_2$ we obtain the value

$$\begin{aligned}
&\left\{\begin{array}{ccc} j_1 & j_2 & j_1+j_2 \\ j_3 & j & j_{23} \end{array}\right\}_q := \left\{\begin{array}{ccc} j_1 & j_2 & j_1+j_2 \\ j_3 & j & s \end{array}\right\}_q \\
&= (-1)^{j_1+j_2+j_3+j}\sqrt{\frac{[2j_1]_q![2j_2]_q![j_1+j_2+j_3+j+1]_q![j_1+j_2-j_3+j]_q!}{[2j_1+2j_2+1]_q![-j_1-j_2+j_3+j]_q![j_2+j_3+s+1]_q!}}\times \\
&\sqrt{\frac{[s-j_1+j]_q![s-j_2+j_3]_q!}{[j_1+j-s]_q![j_1-j+s]_q![j_1+j+s+1]_q![j_2+j_3-s]_q![j_2-j_3+s]_q!}}.
\end{aligned}$$

The expressions (59) and (60) yield

$$\begin{aligned}
&\sqrt{\varsigma(j_{23}+1)}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}+1 \end{array}\right\}_q - \sqrt{\varsigma(-j_{23}-1)}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array}\right\}_q \\
&= [2j_{23}+2]_q\sqrt{[j_1+j_2-j_{12}]_q[j_1+j_2+j_{12}+1]_q}\left\{\begin{array}{ccc} j_1-\frac12 & j_2-\frac12 & j_{12} \\ j_3 & j & j_{23}+\frac12 \end{array}\right\}_q,
\end{aligned} \tag{83}$$

and

$$\begin{aligned}
&\sqrt{\varsigma(-j_{23}-1)}\left\{\begin{array}{ccc} j_1-\frac12 & j_2-\frac12 & j_{12} \\ j_3 & j & j_{23}+\frac12 \end{array}\right\}_q - \sqrt{\varsigma(j_{23})}\left\{\begin{array}{ccc} j_1-\frac12 & j_2-\frac12 & j_{12} \\ j_3 & j & j_{23}-\frac12 \end{array}\right\}_q \\
&= [2j_{23}+1]_q\sqrt{[j_1+j_2-j_{12}]_q[j_1+j_2+j_{12}+1]_q}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array}\right\}_q,
\end{aligned} \tag{84}$$

respectively, where

$$\begin{aligned}
\varsigma(j_{23}) &= [j_{23}-j_3+j_2]_q[j_{23}+j_2+j_3+1]_q[j_{23}-j_1+j+1]_q[j+j_1+j_{23}+1]_q, \\
\varsigma(-j_{23}-1) &= [j_{23}+j_3-j_2+1]_q[j_2+j_3-j_{23}]_q[j_{23}+j_1-j+1]_q[j+j_1-j_{23}]_q.
\end{aligned}$$

The differentiation formulas (61)–(62) give

$$\begin{aligned}
&[2j_{12}]_q A_q^-\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}-1 \end{array}\right\}_q - [2j_{23}]_q\widetilde{A}_q^-\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}-1 \\ j_3 & j & j_{23} \end{array}\right\}_q - \\
&\Big(\varsigma(j_{23})[2j_{12}]_q + [j_1+j_2+j_{12}+1]_q[2j_{23}]_q\widetilde{\Lambda}(j_{12},j_{23},j_1,j_2)\Big)\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array}\right\}_q = 0
\end{aligned} \tag{85}$$

and

$$\begin{aligned}
&[2j_{12}]_q A_q^+\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}+1 \end{array}\right\}_q + [2j_{23}+2]_q\widetilde{A}_q^-\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}-1 \\ j_3 & j & j_{23} \end{array}\right\}_q - \\
&\Big([2j_{12}]_q\varsigma(-j_{23}-1) - [2j_{23}+2]_q[j_1+j_2+j_{12}+1]_q\Big(\widetilde{\Lambda}(j_{12},j_{23},j_1,j_2)+ \\
&\quad [j_1+j_2-j_{12}]_q[2j_{12}]_q[2j_{23}+1]_q\Big)\Big)\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array}\right\}_q = 0,
\end{aligned} \tag{86}$$

respectively, where $A_q^{\pm}$ are given by (73), $\widetilde{A}_q^{\pm}$ by (77) and

$$\begin{aligned}
\widetilde{\Lambda}(j_{12},j_{23},j_1,j_2) =&\varsigma\left(\frac{j_{12}-j_1-j_2}{2}-1\right) - \varsigma\left(\frac{j_{12}-j_1-j_2}{2}\right) - \\
&[2j_{12}]_q\left[j_{23}+\frac{j_1+j_2-j_{12}}{2}\right]_q\left[j_{23}+\frac{j_1+j_2-j_{12}}{2}+1\right]_q.
\end{aligned}$$

If we now use the hypergeometric representations (52) and (54) we obtain two new representations of the $6j$-symbols in terms of the q-hypergeometric function (28)

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = (-1)^{j_{12}+j_3+j} \frac{[2j_2]_q![j_1+j_2-j_3+j]_q!}{[j_3-j_2-j_1+j]_q!} \times$$
$$\sqrt{\frac{[j-j_1+j_{23}]_q![j_3-j_2+j_{23}]_q!}{[j_1+j+j_{23}+1]_q![j_1+j-j_{23}]_q![j_2-j_3+j_{23}]_q![j_2+j_3+j_{23}+1]_q![j_1-j+j_{23}]_q!}} \times$$
$$\sqrt{\frac{[j_3-j_{12}+j]_q![j_{12}+j_3-j]_q![j_3+j_{12}+j+1]_q![j_1-j_2+j_{12}+1]_q!}{[j_3+j_2-j_{23}]_q![j_1+j_2-j_{12}]_q![j_2-j_1+j_{12}]_q![j_{12}-j_3+j]_q![j_1+j_2+j_{12}+1]_q!}} \times$$
$${}_4\mathrm{F}_3 \left(\begin{array}{c} j_{12}-j_1-j_2, -j_1-j_2-j_{12}-1, j_3-j_2-j_{23}, j_{23}+j_3-j_2+1 \\ -2j_2,\ j_3-j_1-j_2+j+1,\ j_3-j_1-j_2-j \end{array} \middle| q\,,1 \right),$$

and

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = (-1)^{j_1+j_{23}+j} \frac{[2j_2]_q![j_1+j_2+j_3+j]_q![j_1+j_2+j_3-j]_q!}{\sqrt{[j_3+j_2-j_{23}]_q![j_1+j_2-j_{12}]_q!}} \times$$
$$\sqrt{\frac{[j-j_1+j_{23}]_q![j_3-j_2+j_{23}]_q!}{[j_1+j+j_{23}+1]_q![j_1+j-j_{23}]_q![j_2-j_3+j_{23}]_q![j_2+j_3+j_{23}+1]_q![j_1-j+j_{23}]_q!}} \times$$
$$\sqrt{\frac{[j_{12}-j_3+j]_q![j_{12}+j_1-j_2+1]_q!}{[j_3-j_{12}+j]_q![j_{12}+j_3-j]_q![j_1+j_2+j_{12}+1]_q![j_2-j_1+j_{12}]_q![j_3+j_{12}+j+1]_q!}} \times$$
$${}_4\mathrm{F}_3 \left(\begin{array}{c} j_{12}-j_1-j_2, -j_1-j_2-j_{12}-1, -j_3-j_2-j_{23}-1, j_{23}-j_3-j_2 \\ -2j_2,\ -j_1-j_2-j_3-j,\ j-j_1-j_2-j_3 \end{array} \middle| q\,,1 \right).$$

Notice that from the above representations the values (74) and (75) also follows. Obviously the above formulas give another two alternative explicit formulas for computing the $6j$-symbols. Finally, from (57)

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = \sqrt{\frac{[j_{23}+j_1+j+1]_q![j_1-j_{23}+j]_q![j_2-j_3+j_{23}]_q![j_2+j_3+j_{23}+1]_q!}{[j_{23}-j_2+j_3]_q![-j_1+j_{23}+j]_q![-j_{12}+j_3+j]_q![j_1+j_2+j_{12}+1]_q!}} \times$$
$$\sqrt{\frac{[j_1+j_{23}-j]_q![j_2+j_3-j_{23}]_q![j_1+j_2-j_{12}]_q![-j_1+j_2+j_{12}]_q!}{[-j_3+j+j_{12}]_q!^{-1}[j_{12}+j_3+j+1]_q!^{-1}[j_1-j_2+j_{12}]_q!^{-1}}} \times$$
$$\sum_{l=0}^{j_1+j_2-j_{12}} \frac{(-1)^{l+j_1+j_2+j_3+j}[2j_{23}+2l-j_1-j_2+j_{12}+1]_q}{[l]_q![j_1+j_2-j_{12}-l]_q![2j_{23}+l+1]_q![j_{23}-j_1-j_2+j_{12}+l+j_2-j_3]_q![j_2+j_3-j_{23}-l]_q!} \times$$
$$\frac{[2j_{23}+l-j_1-j_2+j_{12}]_q![j_{23}+l-j_2+j_3]_q![-j_1+j+j_{23}+l]_q!}{[l+j_{12}-j_2+j+j_{23}+1]_q![j_1+j-j_{23}-l]_q![-j_1+j_{12}+j_3+j_{23}+l+1]_q![-j_2+j_{12}-j+j_{23}+l]_q!},$$

To conclude this section, let us point out that the orthogonality relations (68) and (69) lead to the orthogonality relations for the alternative Racah polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ (52) and their *duals* $\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q$ (64), respectively, as well as the relation (67) between q-Racah and dual q-Racah corresponds to the symmetry property (70).

3.4 Connection between $\widetilde{u}_k^{\alpha,\beta}(x(s),a,b)_q$ and $u_n^{\alpha,\beta}(x(s),a,b)_q$

Let us obtain a formula connecting the two families $\widetilde{u}_k^{\alpha,\beta}(x(s),a,b)_q$ and $u_n^{\alpha,\beta}(x(s),a,b)_q$. In fact, Eqs. (71) and (82) sugest the following relation between both Racah polynomials $\widetilde{u}_k^{\alpha,\beta}(x(s),a,b)_q$ and $u_n^{\alpha,\beta}(x(s),a,b)_q$

$$\widetilde{u}_{b-a-1-n}^{\alpha,\beta}(x(s),a,b)_q = (-1)^{s-a-n} \times$$
$$\frac{\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(b+\alpha-s)\widetilde{\Gamma}_q(b+\alpha+1+s)\widetilde{\Gamma}_q(a+b-\beta-n)}{\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(\alpha+1+n)\widetilde{\Gamma}_q(\beta+1+n)\widetilde{\Gamma}_q(a+b+\alpha+1+n)}\, u_n^{\alpha,\beta}(x(s),a,b)_q. \tag{87}$$

To prove it is sufficient to substitute the above formula in the difference equation (4) of the $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ polynomials. After some straightforward computations the resulting difference equation becomes into the corresponding difference equation for the polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$.

Notice that from this relation follows that

$$
{}_4\mathrm{F}_3\left(\begin{array}{c} a-b+n+1, a-b-\alpha-\beta-n, a-s, a+s+1 \\ a-b+1, 2a-\beta+1, a-b-\alpha+1 \end{array}\,\middle|\, q\,,\,1\right)
$$
$$
= \frac{(\beta+1|q)_{s-a}(b+\alpha+a+1|q)_{s-a}}{(2a-\beta+1|q)_{s-a}(a-b-\alpha+1)_{s-a}}\,{}_4\mathrm{F}_3\left(\begin{array}{c} -n, \alpha+\beta+n+1, a-s, a+s+1 \\ a-b+1, \beta+1, a+b+\alpha+1 \end{array}\,\middle|\, q\,,\,1\right).
$$

This yield to the following identity for terminating ${}_4\phi_3$ basic series, $n, N-n-1, k = 0, 1, 2, \ldots$,

$$
{}_4\varphi_3\left(\begin{array}{c} q^{n-N+1}, q^{-n-N+1}A^{-1}B^{-1}, q^{-k}, q^{-k}D \\ q^{1-N}, q^{-2k}DB^{-1}, q^{1-N}A^{-1} \end{array}\,\middle|\, q\,,\,q\right)
$$
$$
= \frac{q^{-kN}}{A^kB^k}\frac{(qB;q)_k(q^{N-2k}DA;q)_k}{(q^{-2k}DB^{-1};q)_k,(q^{1-N}A^{-1};q)_k}\,{}_4\varphi_3\left(\begin{array}{c} q^{-n}, ABq^n, q^{-k}, q^{-k}D \\ q^{1-N}, qB, q^{N-2k}DA \end{array}\,\middle|\, q\,,\,q\right).
$$

4 Conclusions

Here we have provided a detailed study of two kind of Racah q-polynomials on the lattice $x(s) = [s]_q[s+1]_q$ and also their comparative analysis with the Racah coefficients or $6j$-symbols of the quantum algebra $U_q(su(2))$.

To conclude this paper we will briefly discuss the relation of these q-Racah polynomials with the representation theory of the quantum algebra $U_q(su(3))$. In [20, §5.5.3] was shown that the transformation between two different bases (λ, μ) of the irreducible representation of the classical (not q) algebra $su(3)$ corresponding to the reductions $su(3) \supset su(2)\times u(1)$ and $su(3) \supset u(1)\times su(2)$ of the $su(3)$ algebra in two different subalgebras $su(2)$ is given in terms of the Weyl coefficients that are, up to a sign (phase), the Racah coefficients of the algebra $su(2)$. The same statement can be done in the case of the quantum algebra $su_q(3)$ [5, 18]: The Weyl coefficients of the transformation between two bases of the irreducible representation (λ, μ) corresponding to the reductions $su_q(3) \supset su_q(2)\times u_q(1)$ and $su_q(3) \supset u_q(1)\times su_q(2)$ of the quantum algebra $su_q(3)$ in two different quantum subalgebras $su_q(2)$ coincide (up to a sign) with the q-Racah coefficients of the $su_q(2)$.

In fact, the Weyl coefficients satisfy certain difference equations that are equivalent to the differentiation formulas for the q-Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ and $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ so, following the idea in [20, §5.5.3] we can assure that the main properties of the q-Racah polynomials are closely related with the representations of the quantum algebra $U_q(su(3))$. Finally, let us point out that the same assertion can be done but with the non-compact quantum algebra $U_q(su(2,1))$. This will be carefully done in a forthcoming paper.

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